Studies on Non–interpolating Semi–Lagrangian Scheme and Numerical Solution to KdV Equation[1]

Chen Jiabin (陈嘉滨) and *Wang Jun* (王 军)
Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing 100029, China
Email: cjb@sgi50s.iap.ac.cn
Received December 29, 1995

ABSTRACT

A new non–interpolating semi–Lagrangian scheme has been proposed, which can eliminate any interpolation, and consequently numerical smoothing of forecast fields. Here the new scheme is applied to KdV equation and its performance is assessed by comparing the numerical results with those produced by Ritchie's scheme (1986). The comparison shows that the non–interpolating semi–Lagrangian scheme appears to have efficiency advantages.

Key words: Non–interpolating, Semi–Lagrangian scheme, KdV equation

I. INTRODUCTION

In numerical weather prediction and climate simulation, the explicit schemes are widely used, which time steps are limited by Courant–Friedrichs–Lewy(CFL) criterion. The semi–implicit schemes remove the CFL criterion associated with gravity wave, and their time steps are roughly four to six times those permitted by the explicit scheme for the corresponding Eulerian models. However, the time steps cannot still further be increased and they are restricted in the CFL criterion associated with advection. The semi–Lagrangian scheme broke from the CFL criterion related to the advection, and computational instability did not arise. This scheme combined with the semi–implicit scheme to produce stable and accurate computational scheme was referred to as 'semi–implicit and semi–Lagrangian scheme' (Robert, 1982). It permits the use of larger time steps which are four to six times those permitted by semi–implicit scheme. However, the efficiency of the semi–Lagrangian scheme is affected by use of interpolation. And interpolation leads to the numerical smoothing of forecast field. Ritchie's non–interpolating scheme (1986) still has interpolation, although it is a simple one.

In this paper based on Ritchie's scheme, a new non–interpolating semi–Lagrangian scheme is constructed, which can eliminate any interpolation including the simple one. The new scheme and Ritchie's one are both applied to KdV equation, and the computational results indicate that the new scheme appears to have efficiency advantages.

II. NON–INTERPOLATING SEMI–LAGRANGIAN SCHEME

Consider the one–dimensional advection equation

$$\frac{\partial F}{\partial t} + U\frac{\partial F}{\partial x} = R, \tag{1}$$

[1] Supported by the National Natural Science Foundation of China (4947266) and LASG at Institute of Atmospheric Physics, Chinese Academy of Sciences.

where U is a constant advecting wind in the x–direction. In the non–interpolating approach, advecting wind U in (1) splits up into two parts by adding and subtracting $(p\Delta x / 2\Delta t)$, giving

$$\frac{\partial F}{\partial t} + \frac{p\Delta x}{2\Delta t}\frac{\partial F}{\partial x} = -U'\frac{\partial F}{\partial x} R(x,t), \tag{2}$$

where $U' = U - (p\Delta x / 2\Delta t)$, and Δx and Δt are the spatial and temporal increments, respectively. The left–hand side represents the total derivative of F following a point moving with speed $(p\Delta x / 2\Delta t)$, so that over two time steps the displacement will be P grid lengths. The residual advection related to U' appears on the right–hand side. About the choice of integer P in the left–hand side of (2), it is different from Ritchie's scheme that P in left–hand side of (2) is required to be an even integer nearest $(x_i - 2U\Delta t)$ illustrated in Fig.1. After applying the centered approach (2) becomes as the following:

$$\frac{F(x_i, t+\Delta t) - F(x_i - P\Delta x, t - \Delta t)}{2\Delta t} = \left(R - U'\frac{\partial F}{\partial x}\right)(x_i - P\Delta x / 2, t). \tag{3}$$

Since $x_i - P\Delta x$ and $x_i - P\Delta x / 2$ and are all grid points, interpolation is not required to evaluate $F(x_i, - P\Delta x, t - \Delta t)$ and the right–hand side of (3). It can be proven (not shown here) that the non–interpolating scheme (3) is stable in computation.

III. NUMERICAL SOLUTIONS TO KdV EQUATION

We explore the numercal solutions to KdV equation by Ritchie's scheme (called CONTROL) and non–interpolating scheme presented in the paper (called NISL). As KdV equation has analytic solutions, we can compare non–interpolating semi–Lagrangian presented previously with Ritchie's non–interpolating scheme. The KdV equation may be written as

$$\frac{du}{dt} + \frac{\partial u}{\partial x} + \varepsilon\frac{\partial^3 u}{\partial x^3} = 0, \tag{4}$$

where $du / dt = \partial u / \partial t + u\partial u / \partial x$ and ε is a constant. The analytic solitary wave solution of the KdV equation is

$$u(x,t) = B + A\,\mathrm{sech}^2\left[\sqrt{\frac{A}{12\varepsilon}}(x - ct)\right],$$

$$c = 1 + B + \frac{A}{3} \text{ and } B = -4\sqrt{3\varepsilon A}\,\mathrm{tgh}\left[\frac{1}{4}\left(\sqrt{\frac{A}{3\varepsilon}}\right)\right], \tag{5}$$

where A and c are the amplitude and velocity of the solitary wave, respectively. The analytic

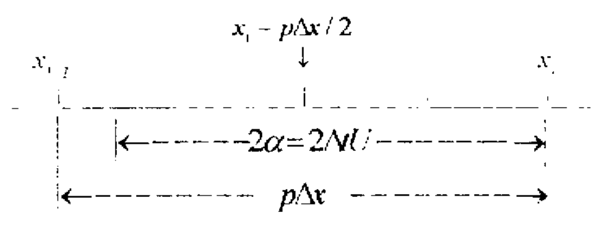

Fig. 1. Schematic diagram illustrating the non–interpolating semi–Lagrangian scheme for the one–dimensional advection problem.

solitary wave solution moves with the constant velocity without changing its wave pattern.

After applying the centered semi−Lagrangian approach, KdV equation (4) becomes:

$$\frac{u(x_i,t+\Delta t)-u(x_i-p\Delta x,t-\Delta t)}{2\Delta t}=-(u'\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}+\varepsilon\frac{\partial^3 u}{\partial x^3})(x_i-\frac{p\Delta x}{2},t), \quad (6)$$

where P is the even integer nearest $(2\Delta tu/\Delta x)$; while for Ritchie's scheme P is the integer nearest $(2t\Delta u/\Delta x)$. And

$$u'=u-\frac{p\Delta x}{2\Delta t} \quad .$$

Now we solve (6) by spectral method using the number of grid points $N=32$, space step $d=0.03125$, $\varepsilon=0.5e{-}4$ and $B=-0.021909$. We take in the x−direction the periodic boundary condition with the period $D=1$. Let time step $\Delta t=2^{-7}$, thus it is necessary to integrate 256 time steps for $t=2$.

The numerical solutions of the KdV equation for CONTROL and NISL scheme are shown in Fig 2. It is seen from Fig.2 that the solitary wave pattern in CONTROL scheme is getting deformed. The cause for the deformation is that for CONTROL scheme the $\partial u/\partial x$ and $\partial^3 u/\partial x^3$ terms near the wave ridge are not computed accurately (Fig.3), because an appropriate average needs to be taken at neighboring grid points when midway location is between grid points. However, there is no need to take any average for NISL scheme, as midway is always a grid point.

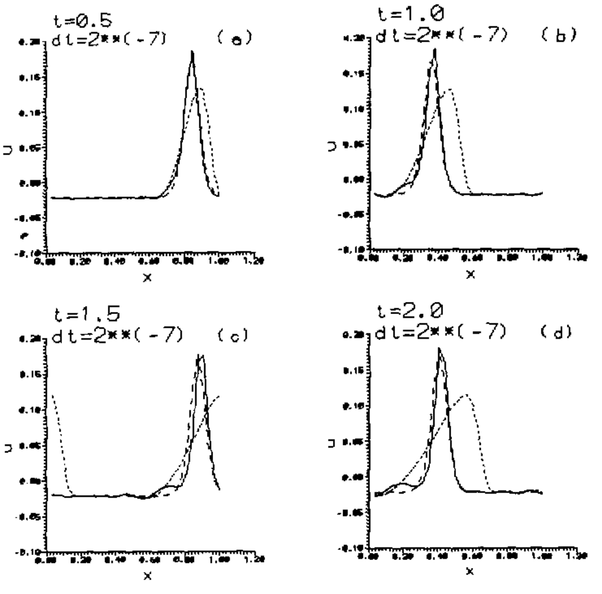

Fig.2. The solution of KdV equation for $dt=2**(-7)$ and (a) $t=0.5$, (b) $t=1.0$, (c) $t=1.5$, (d) $t=2.0$ respectively. The long dashed line shows analytical solution; solid line numerical one by NISL; short dashed line numerical one by CONTROL.

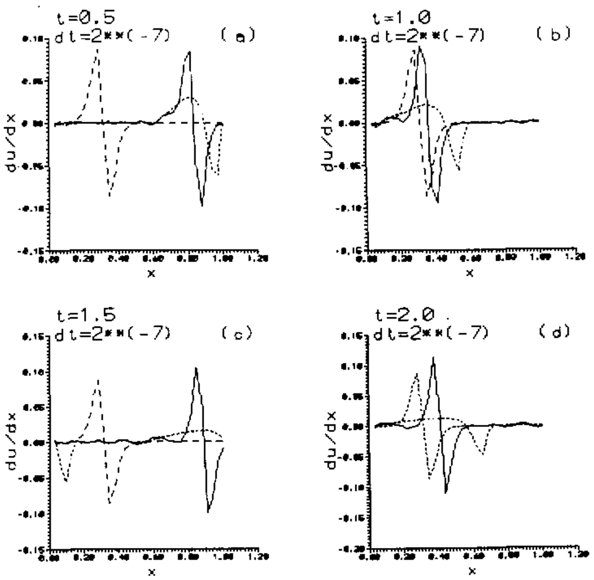

Fig.3. The differentials of the solution for KdV equation (du/dx) at (a) $t=0.5$, (b) $t=1.0$, (c) $t=1.5$, (d) $t=2.0$. The long dashed line is the analytical solution of du/dx, the solid line is the numerical solution of du/dx obtained by NISL, and the short dashed line is the one of du/dx obtained by CONTROL.

IV. CONCLUSIONS

In this paper we have described a non–interpolating semi–Lagrangian scheme,which eliminates any interpolation including a simple one. We have illustrated through the numerical solution to KdV equation the ability of the new scheme to suppress the numerical smoothing caused by interpolation . The differential of the solution for KdV equation is computed by the new scheme more accurately than by CONTROL scheme.

Now the new scheme is applied to the shallow water equations in two dimensions.

REFERENCES

Ritchie, H (1986), Eliminating the interpolation associated with the semi–Lagrangian scheme, *Mon. Wea. Rev.*, **114**: 135–146.

Robert. A (1982), A semi–Lagrangian and semi–implicit numerical integration scheme for the primitive meteorological equation, *J. Meteor. Soc. of Japan*, **60**: 319–325.