A Finite–Mode Model of Ideal Fluid Dynamics on the 2–Sphere

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ABSTRACT

We develop the finite–mode model for a two–dimensional Euler system on the sphere based on Hoppe's discovery in group theory. This model strives to keep as many invariants of the original Euler equation as possible. Theoretically, the number of invariants in this model is limited only by computing power. At present, almost all the popular numerical models in weather and climate researches such as numerical weather prediction models and general circulation models (GCMs) use spectral method. However all these spectrally truncated models do not keep all the invariants except for the energy and the enstrophy. By using this model one is able to study the influence from some other lost invariants. The result from this model is expected to be closer to that of the original Euler equations than from ordinary spectrally truncated models. The relevant fundamental equations and important formulas for this model are given explicitly.

Key words: Energy, Enstrophy, Generalized enstrophies, Euler equations, SU(N) group

I. INTRODUCTION

It is well known that there are an infinite number of invariants in the vorticity equation for an ideal fluid on a two–dimensional manifold M (Arnold and Khesin, 1992). These are any analytic functions of vorticity. Unfortunately this beautiful feature is destroyed by any finite mode truncations. Energy and enstrophy are usually the only constants in a truncated system. Nowadays most numerical simulation models in atmosphere and meteorology such as barotropic models and more complex GCMs are spectrally truncated. Spectral method has been favored over traditional finite difference method in atmospheric predictions for the past decade. It has also been widely used in the other geophysical fluid dynamics and turbulence researches. However one must not forget that on either two–dimensional Cartesian surface or two–dimensional sphere most invariants of an ideal fluid system have been ignored by this popular method. This may be important since we have lost the information provided by the other constants in ideal fluid flow. It will be much more reliable if one can numerically solve a system which can keep as many invariants that are preserved by the original Euler equations as possible when the viscosity in this system is ignored. As the first step in this direction, we concentrate on ideal flows only and develop a finite mode model in this paper. Euler equations have been studied for a long time. Developing a good numerically solvable model for Euler system itself has always been of great value although the real fluid flows in nature and laboratory are absolutely ideal.

This model is based on the fact that the Poisson algebra structure constants (or the structure constants of the area–preserving diffeomorphism group SDiff$M$ whose geodesics are the incompressible fluid flow lines on the manifold M) are the limits of the structure constants of group $SU(N)$ as N tends to infinity after a simple change of normalisation (Hoppe 1989 and Zeitlin, 1991). Hence there are at least N constants in this finite–mode system, namely the energy and the $N-1$ Casimir invariants ("generalized enstrophies"). A first systematic analysis and preliminary numerical analysis of this model on a two–dimensional torus was due to Dowker and Wolski (1992). In this paper we wish to set up the formalities of the corresponding model on the sphere. In comparison with the ordinary spectral model for ideal fluid flows, this model can keep as many invariants as possible theoretically, depending on computing power. Therefore the solution from this numerical model is expected to be closer to that of Euler equations. We will discuss the truncated Euler system and the structure of $SU(N)$ in the next section. In Section 3, the matrix analogue of Euler equations on two–dimensional sphere will be developed. We will prove the constancy of the energy and the generalized enstrophies in this model in Section 4. Some comments and conclusion will be given in Section 5.

II. TRUNCATED EULER SYSTEM AND THE STRUCTURE OF $SU(N)$

Euler's equation on a two–dimensional sphere can be expressed by the stream function ψ or vorticity ζ (Dowker and Wei, 1990). If we choose the coordinates $x_1 = \cos\theta$, $x_2 = \varphi$, then the equation can be written as

$$\frac{\partial \xi}{\partial t} + \{\xi, \psi\} = 0 \tag{1}$$

with

$$\xi = \nabla^2 \psi \ , \tag{2}$$

where $\{\xi,\psi\}$ is the Poisson bracket of ξ and ψ, the Laplacian ∇^2 is given by the standard expression

$$\nabla^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \ , \tag{3}$$

and the eigenvalues are given by

$$\nabla^2 Y_{nm} = -n(n+1)Y_{nm} \ . \tag{4}$$

Here Y_{nm} are spherical harmonics of degree n and order m. The corresponding mode equation is obtained by expanding the stream function and vorticity in terms of spherical harmonics (Wei, 1994), i.e.

$$\psi(\theta,\varphi,t) = \sum_{|m|\leq n} a_{nm}(t) Y_{nm}(\theta,\varphi) \ , \tag{5}$$

which leads to

$$\dot{a}_{nm} = \frac{1}{n(n+1)} \sum_{n'n''} n''(n''+1) a_{n'm'} a_{n''m''} G^{nm}_{n'm'n''m''} \ . \tag{6}$$

Here the dot denotes differentiation with respect to time t, and $G^{nm}_{n'm'n''m''}$ are the Poisson structure constants which are defined by

$$\{Y_{n'm'}, Y_{n''m''}\} = G^{nm}_{n'm'n''m''} Y_{nm} \tag{7}$$

and given explicitly by

$$G^{nm}_{n'm'n''m''} = \oint Y^*_{nm} \left(\frac{\partial Y_{n'm'}}{\partial \cos\theta} \frac{\partial Y_{n''m''}}{\partial \varphi} - \frac{\partial Y_{n'm'}}{\partial \varphi} \frac{\partial Y_{n''m''}}{\partial \cos\theta} \right) d\Omega \ . \tag{8}$$

We are looking for a proper N-dimensional representation of $SU(N)$, whose structure constants $f^{nm}_{n'm'n''m''}$ satisfy

$$\lim_{N\to\infty} f^{nm}_{n'm'n''m''} = G^{nm}_{n'm'n''m''} \ , \tag{9}$$

when a proper normalization is chosen. One has to construct a set of generators T_{nm} of $SU(N)$ in the N-dimensional representation. The generators satisfy the following commutation rule

$$[T_{n'm'}, T_{n''m''}] = c f^{nm}_{n'm'n''m''} T_{nm} \ , \tag{10}$$

where c is a constant depending on the representation. It is known that the spherical harmonics $Y_{nm}(\theta,\varphi)$ can be written in terms of the harmonic homogenous polynomials in the three Cartesian coordinates x_1, x_2 and x_3, i.e.

$$r^n Y_{nm}(\theta,\varphi) = \sum b^{(nm)}_{i_1\cdots i_n} x_{i_1} \cdots x_{i_n} \tag{11}$$

with $b^{(nm)}_{i_1\cdots i_n}$ being totally symmetric and traceless tensors, and

$$x = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta) \ ,$$

$$r^2 = x_1^2 + x_2^2 + x_3^2 \ .$$

Corresponding to each Y_{nm} represented by (11), we replace the x_i by the generators S_i of a suitably normalized N-dimensional representation of $SO(3)$.

$$[S_i, S_j] = \frac{2i}{\sqrt{N^2-1}} \varepsilon_{ijk} S_k, \quad S_1^2 + S_2^2 + S_3^2 = 1 \ , \tag{12}$$

so that we have

$$T_{nm} = \sqrt{\frac{N^2-1}{4}} \sum b^{(nm)}_{i_1\cdots i_n} S_{i_1} \cdots S_{i_n} \ . \tag{13}$$

The dimension of the representation is chosen such that the T_{nm} with $n \leqslant N-1$ form a complete set of traceless $N \times N$ matrices. Therefore, the T_{nm} are the generators of the $SU(N)$ in the chosen representation. The hermiticity of the T_{nm} follows froms the phase convention adopted for the spherical harmonics such that

$$T^+_{nm} = (-1)^m T_{n-m} \ . \tag{14}$$

The tracelessness of the T_{nm} is due to the fact that the tensors $b^{(nm)}_{i_1\cdots i_n}$ are traceless. The relation between the $G^{nm}_{n'm'n''m''}$ and the $f^{nm}_{n'm'n''m''}$ in (9) was proved by Hoppe (1989). Under the defined representation, one has $c = i$ in (10) and

$$f^{nm}_{n'm'n''m''} = \frac{(-1)^m i}{\sqrt{4\pi}} \sqrt{(2n+1)(2n'+1)(2n''+1)} (-1)^N$$

$$\frac{R_N(n')R_N(n'')}{R_N(n)}\begin{bmatrix} n' & n'' & n \\ m' & m'' & -m \end{bmatrix} \begin{Bmatrix} n' & n'' & n \\ s & s & s \end{Bmatrix} \tag{15}$$

where $s = \frac{N-1}{2}$, $\begin{bmatrix} \cdots \\ \cdots \\ \cdots \end{bmatrix}$ and $\begin{Bmatrix} \cdots \\ \cdots \\ \cdots \end{Bmatrix}$ are the conventional $3-j$ and $6-j$ symbols in angular momentum theory (Jones, 1985), and

$$R_N(n) = \sqrt{\frac{(N+n)!(N^2-1)(1-n)}{(N-n-1)!}} \tag{16}$$

There are the same restrictions on $n's$ and $m's$ for the $SU(N)$ structure constants $f^{nm}_{n'm'n''m''}$ as those for $G^{nm}_{n'm'n''m''}$. For example, they are only different from zero if $n+n'+n''$ is odd and $m+m'+m''=0$.

III. MATRIX ANALOGUE OF EULER EQUATION ON S^2

Before we derive the matrix analogue of the Euler equation on S^2, we need to examine T_{nm}, the generators of $SU(N)$. Based on the representation chosen in Section 2, a series of relations or expressions concerning the generators T_{nm} will be derived in this section. The standard $3-j$ and $6-j$ symbols are involved frequently in the following calculations. The explicit form of the generators T_{nm} depends on the representation. For representation (13), the elements of T_{nm} are given by

$$(T_{nm})_{m_1m_2} = (-)^{\frac{3(N-1)}{2}-m_1+1}\sqrt{\frac{2n+1}{16\pi}}R_N(n)\begin{bmatrix} s & n & s \\ -m_1 & m & m_2 \end{bmatrix}. \tag{17}$$

Considering (14) and using (17), one has

$$(T^+_{nm}T_{nm})_{m_1m_2} = (-1)^m\sum_{m_3}(T_{n-m})_{m_1m_3}(T_{nm})_{m_3m_2}$$

$$= (-1)^s R^2_N(n)\begin{bmatrix} s & n & s \\ -m-m_1 & m & m_1 \end{bmatrix}\delta_{m_1,m_2}, \tag{18}$$

so that

$$T^+_{nm}T_{nm} \propto 1 . \tag{19}$$

Due to (10), (17), we have

$$T_{n'm'}T_{n''m''} = h^{nm}_{n'm'n''m''}T_{nm}, \tag{20}$$

where the coefficients $h^{nm}_{n'm'n''m''}$ are given

$$h^{nm}_{n'm'n''m''} = \frac{(-1)^{N+m+1}}{\sqrt{16\pi}} \sqrt{(2n+1)(2n'+1)(2n''+1)} \frac{R_N(n')R_N(n'')}{R_N(n)}$$

$$\times \begin{bmatrix} n' & n'' & n \\ m' & m'' & -m \end{bmatrix} \begin{Bmatrix} n' & n'' & n \\ s & s & s \end{Bmatrix} . \tag{21}$$

With the above formulas for the generators T_{nm}, we can derive fruther useful expressions, such as the trace of the generators:

$$TrT_{nm} = \sum_{m'} (-1)^{3s-m'+1} \sqrt{\frac{2n+1}{16\pi}} R_N(n) \begin{bmatrix} s & n & s \\ -m' & m & m' \end{bmatrix}$$

$$= (-1)^N \sqrt{\frac{N^2(N^2-1)}{16\pi}} \delta_{n,0}\delta_{m,0} . \tag{22}$$

From (20), (21) and (22), it follows that

$$Tr(T_{nm}T_{n'm'}) = h^{n''m''}_{nmn'm'} TrT_{n''m''}$$

$$= \frac{(-1)^{N-m}}{16\pi} R^2_N(n)\delta_{n,n'}\delta_{m+m',0} . \tag{23}$$

We now turn to the finite-mode equation which is expected to approximate the truncated Euler equation (6). Let $w(t)$ and $v(t)$ be two time-dependent elements of $SU(N)$, so that $w^+ = w$ and $v^+ = v$. They can be expanded in terms of the Lie algebra generators T_{nm}

$$w(t) = \sum_{nm} w_{nm}(t)T_{nm} , \quad v(t) = \sum_{nm} v_{nm}(t)T_{nm} . \tag{24}$$

The hermiticity of the coefficients follows from (14), so that

$$w^*_{nm} = (-1)^m w_{n-m} , \quad v^*_{nm} = (-1)^m v_{n-m} . \tag{25}$$

Let us consider the analogue of Euler equation (1),

$$\dot{w} + \alpha[w,v] = 0 \tag{26}$$

with

$$w = \overline{\nabla}^2 v . \tag{27}$$

We identify w and v as the vorticity and the stream (function) matrices repectively. α is a constant that will be specified later. $\overline{\nabla}^2$ is the Lie algebra analogue of the Laplacian, which is chosen to satisfy

$$\overline{\nabla}^2 T_{nm} = -n(n+1)T_{nm} . \tag{28}$$

The expression for $\overline{\nabla}^2$ will be given later. With (27) and (28), it then follows that

$$w_{nm} = -n(n+1)v_{nm} . \tag{29}$$

By inserting all the above relations into (26), with the structure equation (10) of $SU(N)$, one obtains the analogous spectral equation

$$\dot{v}_{nm} = \frac{\alpha i}{n(n+1)} \sum_{n'n''} n''(n''+1) v_{n'm'} v_{n''m''} f^{nm}_{n'm'n''m''} \quad . \tag{30}$$

Thanks to Hoppe's discovery (9) (Hoppe, 1989), if we choose $\alpha = -i$, it is easy to see that

$$\lim_{N\to\infty} \text{Eq.}(30) \longrightarrow \text{Eq.}(6) \ . \tag{31}$$

In order to get the Lie algebra analogue of the Laplacian, we notice that

$$\{Y_{11}, Y_{nm}\} = -i\sqrt{\frac{3}{8\pi}}\sqrt{(n-m)(n+m+1)}\, Y_{n,m+1} \ , \tag{32}$$

$$\{Y_{1-1}, Y_{nm}\} = i\sqrt{\frac{3}{8\pi}}\sqrt{(n+m)(n-m+1)}\, Y_{n,m-1} \ , \tag{33}$$

$$\{Y_{10}, Y_{nm}\} = im\sqrt{\frac{3}{4\pi}}\, Y_{nm} \ . \tag{34}$$

One can then verify that

$$\{\{Y_{11}, Y_{nm}\}, Y_{1-1}\} + \{\{Y_{1-1}, Y_{nm}\}, Y_{11}\} - \{\{Y_{10}, Y_{nm}\}, Y_{10}\}$$
$$= -\frac{3}{4\pi} n(n+1) Y_{nm} \ . \tag{35}$$

Our purpose is to find out a proper expression for $\overline{\nabla}^2$ which satisfies (28) corresponding to (4). The Laplacian operator on S^2 was given by (3). It is very helpful to compare (7) with (10), the structure equations of Poisson algebra and $SU(N)$ respectively. The relation between the two structure constans is given by (9). Actually, the generators T_{nm} are the analogues of the modes Y_{nm} in the same way that $\overline{\nabla}^2$ is the analyog of ∇^2. Examining (35), we believe that the Lie algebra analogue of the Laplacian $\overline{\nabla}^2$ has the form

$$\overline{\nabla}^2 T_{nm} = -\frac{4\pi}{3}\{[[T_{11}, T_{nm}], T_{1-1}] + [[T_{1-1}, T_{nm}], T_{11}]$$
$$- [[T_{10}, T_{nm}], T_{10}]\} \tag{36}$$

To check the validity of this expression, we expand the right hand side of (36) by virtue of the structure equation (10), e.g.

$$\frac{3}{4\pi}\overline{\nabla}^2 T_{nm}$$

$$= -[[T_{11}, T_{nm}], T_{1-1}] + [[T_{1-1}, T_{nm}], T_{11}] - [[T_{10}, T_{nm}], T_{10}]$$

$$= \sum_{n'n''} \left(f^{n',m+1}_{11,nm} f^{n''m}_{n'm+1,1-1} T_{n''m} + f^{n',m-1}_{1-1,nm} f^{n''m}_{n'm-1,11} T_{n''m} - f^{n',m}_{10,nm} f^{n''m}_{n'm,10} T_{n''m} \right)$$

$$= \left(f_{11,nm}^{n,m+1} f_{nm+1,1,-1}^{nm} + f_{1-1,nm}^{n,m-1} f_{nm-1,11}^{nm} - f_{10,nm}^{nm} f_{nm,10}^{nm} \right) T_{nm} \quad . \tag{37}$$

If we insert the expression for $f_{n'm',n''m''}^{m}$, (15), into the right hand side of (37), we find that

$$\begin{aligned}
\overline{\nabla}^2 T_{nm} &= -R_N^2(1)(2n+1)^2 \begin{Bmatrix} 1 & n & n \\ s & s & s \end{Bmatrix}^2 \left[\begin{bmatrix} n & n & 1 \\ m & -m-1 & 1 \end{bmatrix}^2 \right. \\
&\quad \left. + \begin{bmatrix} n & n & 1 \\ m-1 & -m & 1 \end{bmatrix}^2 + \begin{bmatrix} n & n & 1 \\ m & -m & 0 \end{bmatrix}^2 \right] T_{nm} \\
&= -R_N^2(1) \frac{n(n+1)}{N(N^2-1)} T_{nm} \\
&= -n(n+1) T_{nm} \quad .
\end{aligned} \tag{38}$$

In the above derivation, $R_N^2(1) = N(N^2-1)$ was used, and so were some properties of 3−j and 6−j and their expressions e.g. Jones (1985):

$$\begin{Bmatrix} 1 & n & n \\ s & s & s \end{Bmatrix}^2 = (-1)^{N+n} \sqrt{\frac{n(n+1)}{(N-1)(N+1)N(2n+1)}}$$

$$\begin{bmatrix} n & n & 1 \\ m & -m-1 & 1 \end{bmatrix} = (-1)^{n-m} \sqrt{\frac{(n-m)(n+m+1)}{2n(n+1)(2n+1)}}$$

$$\begin{bmatrix} n & n & 1 \\ m-1 & -m & 1 \end{bmatrix} = (-1)^{n-m+1} \sqrt{\frac{(n-m+1)(n+m)}{2n(n+1)(2n+1)}}$$

$$\begin{bmatrix} n & n & 1 \\ m & -m & 0 \end{bmatrix} = (-1)^{n-m} \frac{m}{\sqrt{n(n+1)(2n+1)}}$$

IV. ENERGY AND GENERALIZED ENSTROPHIES

The main goal of this model is to develop a finite−dimensional $SU(N)$ equation which retains more constants of the motion (at least N) and tends to the two−dimensional Euler's equation on the sphere in the infinite N limit. By design, there are N constants for the new system, corresponding to the energy and the $N-1$ generalized enstrophies. We suppose that N is odd, so that $s = \frac{N-1}{2}$ is an integer. Before we discuss the conservation properties, we need the fact that $\overline{\nabla}^2$ is Hermitian, i.e.

$$Tr(d^+ \overline{\nabla}^2 e) = Tr(\overline{\nabla}^2 d^+ e) \ , \tag{39}$$

where d and e are any elements of $SU(N)$, and of course

$$d^+ = d, \quad e^+ = e \ .$$

It is not difficult to see

$$Tr(d[[T_{11},e],T_{1-1}]) = Tr([[T_{11},d],T_{1-1}]e) \ . \tag{40}$$

According to the definition of $\overline{\nabla}^2$ (36), (40) leads to

$$Tr(d\overline{\nabla}^2 e) = Tr(\overline{\nabla}^2 de) \ , \tag{41}$$

which is equivalent to (39). We note that the finite analogue of integration over S^2 is the trace. One can define the energy and enstrophies as

$$E = \frac{1}{2} Tr(wv) \tag{42}$$

and

$$\Omega_k = \frac{1}{2} Tr(w^k) \ , \tag{43}$$

where $k = 2 \cdots N$, and $\Omega_1 = 0$. E and Ω_k are constants of the system,

$$\frac{dE}{dt} = 0, \quad \frac{d\Omega_k}{dt} = 0 \ . \tag{44}$$

Let us first look at the energy. From the definition (42), one has

$$\frac{dE}{dt} = \frac{1}{2} Tr(\dot{w}v + w\dot{v}) \ . \tag{45}$$

With (39) and $w = \overline{\nabla}^2 v$, the second term of the right hand side of (45) can be reexpressed as

$$Tr(w\dot{v}) = Tr(\overline{\nabla}^2 v\dot{v}) = Tr(v\dot{w}) \tag{46}$$

so that

$$\frac{dE}{dt} = Tr(\dot{w}v) = Tr(i[w,v]v) = 0 \ . \tag{47}$$

The second step in above equation comes from using equation (26). Similarly, we can prove the second conservation of (44). e.g.

$$\frac{d\Omega_k}{dt} = \frac{1}{2} Tr(kw^{k-1}\dot{w}) = \frac{1}{2} kiTr(w^{k-1}[w,v]) = 0 \ . \tag{48}$$

We next turn to the detailed expressions for the constants in terms of the components of the stream function. In light of the definitions of the energy and the enstrophies (42) and (43), we have

$$\begin{aligned} E &= \frac{1}{2} Tr(wv) \\ &= \frac{1}{2} \sum_{nn'} -n(n+1) v_{nm} v_{n'm'} Tr(T_{nm} T_{n'm'}) \\ &= \frac{1}{32\pi} \sum_{n} n(n+1) R_N^2(n) |v_{nm}|^2 \ . \end{aligned} \tag{49}$$

Formula (23) was used in the above derivation. Similary for the generalized enstrophies, we have

$$
\begin{aligned}
\Omega_k &= -\frac{1}{2} Tr(w^k) \\
&= \frac{(-1)^k}{2} \sum_{n_1 \cdots n_k} n_1(n_1+1) \cdots n_k(n_k+1) \cdot v_{n_1 m_1} \cdots v_{n_k m_k} Tr(T_{n_1 m_1} \cdots T_{n_k m_k}) \\
&= \frac{(-1)^{(k+1)(s+1)}}{2(16\pi)^{k/2}} \sum_{n_1 \cdots n_k} (-1)^{m_k} + \sum_{i=1}^{k-2} (k-i-1) m_i \prod_{j=1}^{k} R_N(n_j) n_j (n_j+1) \sqrt{2n_j+1} \\
&\quad \cdot v_{n_j m_j} \cdot \sum_m (-1)^{(k+1)m} \begin{bmatrix} n_1 & s & s \\ m_1 & m & -m-\mu_1 \end{bmatrix} \begin{bmatrix} n_2 & s & s \\ m_2 & m+\mu_1 & -m-\mu_2 \end{bmatrix} \\
&\quad \cdots \begin{bmatrix} n_{k-1} & s & s \\ m_{k-1} & m+\mu_{k-2} & -m-\mu_{k-1} \end{bmatrix} \begin{bmatrix} n_k & s & s \\ m_k & m+\mu_{k-1} & -m \end{bmatrix} \\
&= \frac{(-1)^{(k+1)(s+1)}}{2(16\pi)^{k/2}} \sum_{n_1 \cdots n_k} (-1)^{m_k + \sum_{i=1}^{k-2} (k-i-1) m_i} \prod_{j=1}^{k} R_N(n_j) n_j (n_j+1) \\
&\quad \sqrt{2n_j+1}\, v_{n_j m_j} \sum_m (-1)^{(k+1)m} \prod_{p=1}^{k} \begin{bmatrix} n_p & s & s \\ m_p & m+\mu_{p-1} & -m-\mu_p \end{bmatrix}
\end{aligned}
\tag{50}
$$

where $\mu_j \equiv \sum_{i=1}^{j} m_i$, and the formula of $Tr(T_{n_1 m_1} T_{n_2 m_2} \cdots T_{n_k m_k})$ has been used. The derivation will be given in the appendix.

V. COMMENTS AND CONCLUSION

Perhaps the most interesting feature of two–dimensional Euler system is that the flow preserves an infinite number of quantities that are dynamically relevant. However, any finite mode truncation fails to retain this property. In this sense, we lose the information given by other constants except for the energy and the enstrophy which do remain constant. People have been trying to overcome this difficulty, and we have seen that different theories have appeared emphasizing different aspects, e.g. the recent development on two–dimensional torus by Robert and Sommeria (1991) and Miller et al. (1992). Nevertheless, all the models are simplified one way or another and based on assumptions which need further justification. None of them are universally accepted so far. Each model has its own advantages and disadvantages. It is hard to judge which theory is the best at present. From our point of view, the final justification of a model has to rest on the comparisons with the results from simulations and experiments.

Most work on the two–dimensional Euler system has been in periodic Cartesian geometry (e.g. Batchelor, 1967; Kraichnan and Montgomery, 1980; Kraichnan, 1975). A finite–mode model on a torus was developed by Dowker and Wolski (1992) who gave the first systematic anlysis and detailed description, including some numerical tests. There is no doubt that practical calculations on a torus are simpler than on a sphere due to the topological and metric difference. The model developed in this paper may include a rotation term which is usually used in geophysical fluid dynamics. In spherical geometry, two-dimensional ideal hydrodynamics can model large scale, long time atmospheric flows (over 100 km, e.g. Kraichnan and Montgomery, 1980) and some other geophysical flows (e.g. Holloway, 1986). The ultimate advantage of this model is that the truncated system on the two-dimensional sphere has at least N invariants. Needless to say, the more invariants are included in the numerical model, the closer this model represents the original Euler equation. In turbulence researches this model may be of help as well. It will be interesting to analyze the transfers of energy, enstrophy and generalized enstrophies between different scales.

We have set up the formal structure for the kinematics and dynamics of this model. The next step would be to numerically integrate the matrix equations of the motion. These equations can be solved easier than those in the other theories, even though we are dealing with ideal hydrodynamic equations on a sphere rather than a torus. In principle, one can have as many invariants as possible depending on the computer power. This will make it available to analyze the restraints that the other ignored constants may impose on the motions of ideal fluid flows. In mathematical aspect, it is necessary to investigate the nature of the limit $N \to \infty$ to elucidate exactly how the Lie algebra equations turn into the continuum Euler equations. This would necessitate the development of a spherical analogue of the finite Fourier transform used in the torus case. Details may be provided in our next communication. It is our hope that this model will throw light on solving the two-dimensional Euler equations on sphere. We also hope that this model will contribute to the future development of more robust geophysical and atmospheric models.

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Appendix: Calculation of $Tr(T_{n_1 m_1} T_{n_2 m_2} \cdots T_{n_k m_k})$

In this appendix, we derive a detailed expression for $Tr(T_{n_1 m_1} T_{n_2 m_2} \cdots T_{n_k m_k})$ in a form that is both concise and ready for use.

$$
\begin{aligned}
TR &= Tr(T_{n_1 m_1} T_{n_2 m_2} \cdots T_{n_k m_k}) \\
&= \sum_{\nu_2 \cdots \nu_{k-1}} h^{\nu_2 \mu_2}_{n_1 m_1 n_2 m_2} h^{\nu_3 \mu_3}_{\nu_2 \mu_2 n_3 m_3} \cdots h^{\nu_{k-1} \mu_{k-1}}_{\nu_{k-2} \mu_{k-2} n_{k-1} m_{k-1}} Tr(T_{\nu_{k-1} \mu_{k-1}} T_{n_k m_k}) \\
&= \sum_{\nu_2 \cdots \nu_{k-1}} h^{\nu_2 \mu_2}_{n_1 m_1 n_2 m_2} h^{\nu_3 \mu_3}_{\nu_2 \mu_2 n_3 m_3} \cdots h^{\nu_{k-2} \mu_{k-2}}_{\nu_{k-3} \mu_{k-3} n_{k-2} m_{k-2}} h^{\nu_{k-1} \mu_{k-1}}_{\nu_{k-2} \mu_{k-2} n_{k-1} m_{k-1}} \\
&\qquad (-1)^{m_k+1} \delta_{n_k, \nu_{k-1}} \delta_{\nu_k, 0} \frac{R_N^2(n_k)}{16\pi} \\
&= \frac{(-1)^{1+\sum_{i=2}^{k-2} \mu_i}}{(16\pi)^{k/2}} \prod_{i=1}^{k} [R_N(n_i)\sqrt{2n_i+1}] Q
\end{aligned} \tag{51}
$$

where Q is displayed below, and can be simplified further.

$$Q = \prod_{j=2}^{k-2}(2\nu_j+1)\begin{bmatrix} n_1 & n_2 & \nu_2 \\ m_1 & m_2 & -\mu_2 \end{bmatrix}\begin{Bmatrix} n_1 & n_2 & \nu_2 \\ s & s & s \end{Bmatrix}$$

$$\prod_{i=2}^{k-3}\begin{bmatrix} \nu_i & n_{i+1} & \nu_{i+1} \\ \mu_i & m_{i+1} & -\mu_{i+1} \end{bmatrix}\begin{Bmatrix} \nu_i & n_{i+1} & \nu_{i+1} \\ s & s & s \end{Bmatrix}$$

$$\begin{bmatrix} \nu_{k-2} & n_{k-1} & n_k \\ \mu_{k-2} & m_{k-1} & m_k \end{bmatrix}\begin{Bmatrix} \nu_{k-2} & n_{k-1} & n_k \\ s & s & s \end{Bmatrix}. \tag{52}$$

The strategy is to replace the $6-j$'s with $3-j$'s. For example,

$$\begin{bmatrix} n_1 & n_2 & n_3 \\ m_1 & m_2 & m_3 \end{bmatrix}\begin{Bmatrix} n_1 & n_2 & n_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}$$

$$= \sum_{p_1,p_2,p_3}(-1)^S\begin{bmatrix} n_1 & l_2 & l_3 \\ m_1 & p_2 & -p_3 \end{bmatrix}\begin{bmatrix} l_1 & n_2 & l_3 \\ -p_1 & m_2 & p_3 \end{bmatrix}\begin{bmatrix} l_1 & l_2 & n_3 \\ p_1 & -p_2 & m_3 \end{bmatrix}$$

where $S = l_1 + l_2 + l_3 + p_1 + p_2 + p_3$. The orthogonality properties of the $3-j$ symbols then give

$$Q = \prod_{j=2}^{k-2}(2\nu_j+1)\sum_m(-1)^{3s+3m+\nu_1+\nu_2}\begin{bmatrix} n_1 & s & s \\ m_1 & m & -m-m_1 \end{bmatrix}\cdot$$

$$\begin{bmatrix} s & n_2 & s \\ -m-\mu_2 & m_2 & m+\mu_1 \end{bmatrix}\begin{bmatrix} s & s & \nu_2 \\ m+\mu_2 & -m & -\mu_2 \end{bmatrix}$$

$$\prod_{i=2}^{k-3}\sum_{a_i^1 a_i^2 a_i^3}(-1)^{3s+a_i^1+a_i^2+a_i^3}\cdot\begin{bmatrix} \nu_i & s & s \\ \mu_i & a_i^2 & -a_i^3 \end{bmatrix}$$

$$\begin{bmatrix} s & n_{i+1} & s \\ -a_i^1 & m_{i+1} & a_i^3 \end{bmatrix}\begin{bmatrix} s & s & \nu_{i+1} \\ a_i^1 & -a_i^2 & -\mu_{i+1} \end{bmatrix}$$

$$\sum_{a_{k-2}^1 a_{k-2}^2 a_{k-2}^3}(-1)^{3s+a_{k-2}^1+a_{k-2}^2+a_{k-2}^3}\begin{bmatrix} \nu_{k-2} & s & s \\ \mu_{k-2} & a_{k-2}^2 & -a_{k-2}^3 \end{bmatrix}$$

$$\begin{bmatrix} s & n_{k-1} & s \\ -a_{k-2}^1 & m_{k-1} & a_{k-2}^3 \end{bmatrix}\begin{bmatrix} s & s & n_k \\ a_{k-2}^1 & -a_{k-2}^2 & m_k \end{bmatrix} \tag{53}$$

In order to simplify this expression, we have to use the $3-j$ properties fully, and infer the general rules among the indices. For this purpose, we rewrite (53) more clearly as

$$Q = \prod_{j=2}^{k-2}(2\nu_j + 1)\sum_m(-1)^{3s+3m+\nu_1+\nu_2}\begin{bmatrix} n_1 & s & s \\ m_1 & m & -m-m_1 \end{bmatrix}$$

$$\begin{bmatrix} n_2 & s & s \\ m_2 & m+\mu_1 & -m-\mu_2 \end{bmatrix}\begin{bmatrix} s & s & \nu_2 \\ m+\mu_2 & -m & -\mu_2 \end{bmatrix}$$

$$\sum_{a_2^1 a_2^2 a_2^3}(-1)^{3s+a_2^1+a_2^2+a_2^3}\begin{bmatrix} s & s & \nu_2 \\ a_2^3 & -a_2^2 & -\mu_2 \end{bmatrix}$$

$$\begin{bmatrix} n_3 & s & s \\ m_3 & a_2^3 & -a_2^1 \end{bmatrix}\begin{bmatrix} s & s & \nu_3 \\ a_2^1 & -a_2^2 & -mu_3 \end{bmatrix}\cdots$$

$$\sum_{a_{k-3}^1 a_{k-3}^2 a_{k-3}^3}(-1)^{3s+a_{k-3}^1+a_{k-3}^2+a_{k-3}^3}\begin{bmatrix} s & s & \nu_{k-3} \\ a_{k-3}^3 & -a_{k-3}^2 & -\mu_{k-3} \end{bmatrix}$$

$$\begin{bmatrix} n_{k-2} & s & s \\ m_{k-2} & a_{k-3}^3 & -a_{k-3}^1 \end{bmatrix}\begin{bmatrix} s & s & \nu_{k-2} \\ a_{k-3}^1 & -a_{k-3}^2 & -mu_{k-2} \end{bmatrix}$$

$$\sum_{a_{k-2}^1 a_{k-2}^2 a_{k-2}^3}(-1)^{3s+a_{k-2}^1+a_{k-2}^2+a_{k-2}^3}\begin{bmatrix} s & s & \nu_{k-2} \\ a_{k-2}^3 & -a_{k-2}^2 & -\mu_{k-2} \end{bmatrix}$$

$$\begin{bmatrix} n_{k-1} & s & s \\ m_{k-1} & a_{k-2}^3 & -a_{k-2}^1 \end{bmatrix}\begin{bmatrix} n_k & s & s \\ m_k & -a_{k-2}^1 & -a_{k-2}^2 \end{bmatrix}. \tag{54}$$

In above expressions, only the $3-j$ symbols are involved, and the orthogonality of the $3-j$'s can be applied, i.e.

$$\sum_{n_3 m_3}(2n_3+1)\begin{bmatrix} n_1 & n_2 & n_3 \\ m_1 & m_2 & m_3 \end{bmatrix}\begin{bmatrix} n_1 & n_2 & n_3 \\ p_1 & p_2 & m_3 \end{bmatrix} = \delta_{m_1,p_1}\delta_{m_2,p_2} \tag{55}$$

which will simplify the expressions greatly. The relations between the indices can then be deduced, so that

$$a_2^3 = \mu_2 + m, \quad a_2^2 = m, \quad a_2^1 = \mu_3 + m \tag{56}$$

with

$$a_2^1 + a_2^2 + a_2^3 = 3m + \mu_2 + \mu_3 \ . \tag{57}$$

For the same reason, the relation between more general indices can be established, such that

$$a_{k-2}^3 = \mu_{k-2} + m, \quad a_{k-2}^2 = m, \quad a_{k-2}^1 = \mu_{k-1} + m \tag{58}$$

together with

$$a^1_{k-2} + a^2_{k-2} + a^3_{k-2} = 3m + \mu_{k-2} + \mu_{k-1} \quad . \tag{59}$$

Consequently, (52) can be rewritten as

$$Q = \sum_m (-1)^H \begin{bmatrix} n_1 & s & s \\ m_1 & m & -m-\nu_1 \end{bmatrix} \begin{bmatrix} n_2 & s & s \\ m_2 & m+\mu_1 & -m-\nu_2 \end{bmatrix} \cdots$$

$$\begin{bmatrix} n_{k-1} & s & s \\ m_{k-1} & m+\mu_{k-2} & -m-\nu_{k-1} \end{bmatrix} \begin{bmatrix} n_k & s & s \\ m_k & m+\mu_{k-1} & -m \end{bmatrix}$$

$$= \sum_m (-1)^H \prod_{p=1}^{k} \begin{bmatrix} n_p & s & s \\ m_p & m+\mu_{p-1} & -m-\mu_p \end{bmatrix} \tag{60}$$

where H can be simplified, with the even integers ignored

$$\begin{aligned} H &= (3s+3m)(k-3) + \mu_1 + \mu_2 + \mu_2 + \mu_3 + \cdots + \mu_{k-1} \\ &= 3s(k-3) + 3(k-3)m + \mu_1 + \mu_{k-1} + 2\sum_{i=2}^{k-2} \mu_i \\ &\Rightarrow s(k+1) + (k+1)m + m_1 + m_k \quad . \end{aligned} \tag{61}$$

Therefore, we have the final formula

$$TR = Tr(T_{n_1 m_1} T_{n_2 m_2} \cdots T_{n_k m_k})$$

$$= \frac{(-1)^{(k+1)s + m_k + 1 + \sum_{i=1}^{k-2} \mu_i}}{(16\pi)^{k/2}} \prod_{i=1}^{k} [R_N(n_i)\sqrt{2n_i+1}]$$

$$\sum_m (-1)^{(k+1)m} \prod_{p=1}^{k} \begin{bmatrix} n_p & s & s \\ m_p & m+\mu_{p-1} & -m-\mu_p \end{bmatrix} \tag{62}$$

REFERENCES

Arnold, V. I. and Khesin, B. A. (1992), Topological methods in hydrodynamics, *Ann. Rev. Fluid Mech.*, **24**: 145–166.

Batchelor, G. K. (1967), *The Theory of Turbulence*, Cambridge University Press, Cambridge, UK, 179 pp.

Dowker, J. S. and Wei, M. (1990), Volume–preserving diffeomorphisms and the stability of the atmosphere, *Class. Quantum Grav.*, **7**: 2361–2365.

Dowker, J. S. and Wolski, A. (1992), Finite model of two–dimensional ideal hydrodynamics, *Physical Review A*, **46**: 6417–6430.

Holloway, G. (1986), Eddies, waves, circulation, and mixing: statistical geofluid mechanics, *Ann. Rev. fluid Mech.*, **18**: 91–147.

Hoppe, J. (1989), Diffeomorphism group, quantization, and SU(∞), *Int. J. Mod. Phys. A*, **4**: 5235–5248. and the references herein.

Jones, M.N.(1985), *Spherical Harmonics and Tensors for Classical Field Theory*, Research Studies Press Ltd. England.

Kraichnan, R. K. (1975), Statistical dynamics of two-dimensional flow, *J. Fluid Mech.*, **67**: 155–175.

Kraichnan, R. K. and Montgomery, D. (1980), Two-dimensional turbulence, *Rep. Prog. Phys.*, **43**: 547–619.

Miller, J., Weichman. P.B. and Cross, M. C. (1992), Statistical mechanics, Eular's equation, and Jupiter's Red Spot, *Phys. Rev. A*, **45(4)**: 2328–2359.

Robert, R. and Sommeria, J. (1991), Statistical equilibrium states for two-dimensional flows, *J. Fluid Mech.*, **229**: 291–310.

Wei, M. (1994), Chaos and Diffeomorphisms in Hydrodynamics on Spheres, Ph.D Thesis, University of Manchester, UK 188 pp.

Zeitlin, V. (1991), Finite-mode analogs of 2D ideal hydrodynamics: coadjoint orbits and local canonical structure, *Physica D*, **49**: 353–362.