# Multispectrum Method and the Computation of Vapor Equation①

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ABSTRACT

In order to improve the practicality of spectral method and the efficiency of computation, the multi-spectrum method is proposed on the basis of multi-grid method. Coarse spectra are used to compute the slow nonlinear part (including physical process), while fine spectra are used to compute the fast linear part. This method not only can reduce computation time, but also can obtain computational efficiency similar to that from only fine spectra. Thus, it is an economical numerical method. Both explicit complete-square-conservation scheme and multispectrum scheme are used to improve IAP $L_9$ $T_{42}$ spectral climate models, with and without physical forcings respectively, and the advantage of reducing computation time is obtained satisfactorily. In order to overcome the difficulty that vapor equation is very sensitive to the change of time step, the square-conservation semi-Lagrangian scheme is used to solve vapor equation. Because the semi-Lagrangian scheme has the property of square-conservation, computational instability can be avoided. When time step becomes longer with the semi-Lagrangian Scheme, through numerical examples, the vapor transportation can be depicted objectively and the effect of precipitation simulation can be modified.

**Key words:** Spectral method, Multispectrum, Semi-Lagrangian method, Computation of vapor

## I. INTRODUCTION

Numerical methods in common use to solve partial differential equation are spectral method, finite-difference method and finite element method. Because of the large computational amount, spectral method is not practical until FFT (fast Fourier transform) is proposed. To meet practical needs, it's necessary to establish time-saving numerical methods suitable for spectral models. Several years ago, explicit complete square-conservation difference schemes and its time-saving techniques were developed and obvious time benefits were obtained (Wang et al., 1990; Ji et al., 1991; Wang et al., 1994; Wang et al., 1993). Recently, another time-saving method i.e. multispectrum method, is proposed on the basis of spectral method (Ji and Wang, 1996). In this paper, multispectrum scheme on short-term climate simulation is applied. Especially for the simulation of precipitation, complete energy conservative semi-Lagrangian method is proposed to solve vapor equation. In Section 2, basic idea of multispectrum method is introduced. In Section 3, some good results and currently existing problems on the application of multispectrum method are proposed. In Section 4, square-conservation semi-Lagrangian method is applied to improve the calculation of vapor equation.

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## II. BASIC IDEA OF MULTISPECTRUM METHOD

For the spherical baroclinic atmospheric equations in $\sigma$-coordinate system:

$$\begin{aligned}
&\frac{du}{dt} - \frac{uv}{a}\mathrm{tg}\theta - fv = -\frac{1}{a\cos\theta}\left[\frac{\partial\varphi}{\partial\lambda} + RT_v\frac{\partial}{\partial\lambda}\ln P_s\right] + P_u \ , \\
&\frac{dv}{dt} - \frac{u^2}{a}\mathrm{tg}\theta + fu = -\frac{1}{a}\left[\frac{\partial\varphi}{\partial\theta} + RT_v\frac{\partial}{\partial\theta}\ln P_s\right] + P_v \ , \\
&\frac{dT}{dt} - \frac{RT_v}{C_pP}w = P_t, \quad \frac{d}{dt}\ln P_s = -D\frac{\partial\dot{\sigma}}{\partial\sigma}, \quad \frac{dq}{dt} = P_q \ , \\
&\frac{\partial\varphi}{\partial\ln\sigma} = -RT_v \ ,
\end{aligned} \tag{1}$$

where

$$\begin{cases}
\dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \dfrac{u}{a\cos\theta}\dfrac{\partial}{\partial\lambda} + \dfrac{v}{a}\dfrac{\partial}{\partial\theta} + \dot{\sigma}\dfrac{\partial}{\partial\sigma} \\
w = \dfrac{dP}{dt} = P_s\dot{\sigma} + \sigma\left(\dfrac{U}{a\cos\theta}\dfrac{\partial P_s}{\partial\lambda} + \dfrac{v}{a}\dfrac{\partial P_s}{\partial\theta}\right) \\
\sigma = \dfrac{P}{P_s}
\end{cases} \tag{2}$$

Setting

$$\begin{cases}
\eta = f + \xi, \quad \xi = \dfrac{1}{a\cos\theta}\left[\dfrac{\partial v}{\partial\lambda} - \dfrac{\partial}{\partial\theta}(u\cos\theta)\right] \\
D = \dfrac{1}{a\cos\theta}\left[\dfrac{\partial u}{\partial\lambda} - \dfrac{\partial}{\partial\theta}(v\cos\theta)\right], \quad E = \dfrac{1}{2}(u^2 + v^2)
\end{cases} \tag{3}$$

then the first two equations in (1) can be reduced as follows:

$$\begin{cases}
\dfrac{\partial u}{\partial t} - \eta v + \dfrac{1}{a\cos\theta}\left[\dfrac{\partial}{\partial\lambda}(\varphi + E) + RT_v\dfrac{\partial}{\partial\lambda}\ln P_s\right] + \dot{\sigma}\dfrac{\partial u}{\partial\sigma} = P_u \ , \\
\dfrac{\partial v}{\partial t} + \eta u + \dfrac{1}{a}\left[\dfrac{\partial}{\partial\theta}(\varphi + E) + RT_v\dfrac{\partial}{\partial\theta}\ln P_s\right] + \dot{\sigma}\dfrac{\partial v}{\partial\sigma} = P_v \ .
\end{cases} \tag{4}$$

Now, (4) can be transformed into vorticity equation and divergence equation. Letting

$$\mu = \sin\theta, \quad U = u\cos\theta, \quad V = v\cos\theta, \quad T' = T - T_\theta, \quad T'_v = T_v - T_\theta \tag{5}$$

and

$$\begin{cases}
\vec{F} = [\eta, D, T', \ln P_s, q] \\
\tilde{L}_1\vec{F} = [\tilde{L}_1^{(\eta)}\eta, \tilde{L}_1^{(D)}D, \tilde{L}_1^{(T)}T', \tilde{L}_1^{(p)}(\ln P_s), \tilde{L}_1^{(q)}q] \\
\tilde{L}_2\vec{F} = [0, \tilde{L}_2^{(D)}D, \tilde{L}_2^{(T)}T', \tilde{L}_2^{(p)}(\ln P_s), 0] \\
\vec{P} = [P_\eta, P_D, P_T, 0, P_q]
\end{cases} \tag{6}$$

then the equations can be written in operator form

$$\frac{\partial\vec{F}}{\partial t} = \tilde{L}_1\vec{F} + \tilde{L}_2\vec{F} + \vec{P} \ . \tag{7}$$

If friction is neglected and adiabatic condition is assumed, then $\vec{P} = 0$ and (7) can be simplified as:

$$\frac{\partial \vec{F}}{Lt} = \tilde{L}_1 \vec{F} + \tilde{L}_2 \vec{F} \ . \tag{8}$$

By analyzing operator equation (8), it is obvious that the right two parts have different physical properties: the first is mainly composed of slow waves and high-frequency effect is very weak, but the second consists of quick waves and high-frequency effect is rather strong. So, when (8) is solved by using spectral method, it is not economical and is necessary to use fine spectra for the first part. It can be computed by using coarse spectra and the computational efficiency will not be affected greatly. By this way, benefits of saving time can be obtained. For the second part, because of strong high-frequency waves in it, fine spectra must be used otherwise the computational efficiency will be greatly damaged. The second part is much simpler than the first part and the computation amount is relatively small, so by using fine spectra the computing time will not increase much. This method, that coarse spectra ( fine spectra) are used to compute slow waves (quick waves) is called multispectrum method. It comes from multigrid method and is the extension to spectral method. It not only can save time, but also can keep good simulation effect and it is theoretically a satisfactory economical numerical method.

## III. RUDIMENTARY NUMERICAL RESULTS AND SOME PROBLEMS

To examine the time-saving property and computational efficiency of the multispectrum method, it is applied to improve the $IAPL_9T_{42}$ dynamic spectral model. Using the (global) observation data on June 8, 1992 as initial field, making 5-day prediction and comparing the results with those from the original model and observations, we can see that the computational efficiency is satisfactory.

**Table 1.** Anomaly Correlation Coefficient and Mean Variance of Global (90°S—90°N) Prediction Results for New and Old Models with Observed Data

Initial data: 92.06.08: 5-day prediction

| Layer | Correlation coefficient (old scheme) | Correlation coefficient (new scheme) | Difference | Variance of old scheme | Variance of new scheme | Difference |
|---|---|---|---|---|---|---|
| 100 hPa | 0.5689 | 0.6044 | 0.0355 | 116.02 | 111.47 | −4.55 |
| 200 hPa | 0.4730 | 0.5234 | 0.0504 | 134.92 | 126.21 | −8.71 |
| 300 hPa | 0.4256 | 0.4879 | 0.0623 | 137.99 | 127.42 | −10.57 |
| 500 hPa | 0.3869 | 0.4429 | 0.0560 | 115.72 | 106.30 | −9.42 |
| 700 hPa | 0.3369 | 0.3823 | 0.0454 | 103.05 | 93.74 | −9.31 |
| 850 hPa | 0.2447 | 0.2793 | 0.0346 | 101.13 | 91.32 | −9.81 |
| Average of 6 layers | 0.4060 | 0.4534 | 0.0474 | 118.14 | 109.41 | −8.73 |

The comparison shows that the predictive effect of new scheme is better than that of the original one. Although it is difficult to detect the minor difference from prognostic charts (omitted), the conclusion can be obtained from their anomaly correlation coefficient and mean variance with observed data (Table 1). The longer the prediction time is, the more obvious the difference exists.

Furthermore, Table 2 shows the evolution of whole kinetic energy, absolute average vorticity and divergence with time. It can be known that the whole property of the new scheme is better than that of the old one, and changes are uniform and stability is better. Es-

pecially, the time step of the new scheme is three times longer than that of the old one, so computing time can be saved.

**Table 2.** Comparison of Kinetic Energy Vorticity and Divergence between New Scheme and Old Scheme

| Time (hour) | Kinetic energy of old scheme (1E+8) | Kinetic energy of new-scheme (1E+8) | Vorticity of old scheme (1E-4) | Vorticity of new scheme (1E-4) | Divergence of old scheme (1E-5) | Divergence of new scheme (1E-5) |
|---|---|---|---|---|---|---|
| 000 | 0.11106 | 0.11106 | 0.20227 | 0.20227 | 0.26864 | 0.26864 |
| 024 | 0.12055 | 0.11635 | 0.21943 | 0.20888 | 0.37728 | 0.31499 |
| 0.48 | 0.12916 | 0.12036 | 0.23527 | 0.21688 | 0.42374 | 0.33959 |
| 0.72 | 0.14019 | 0.12562 | 0.26042 | 0.22891 | 0.52064 | 0.39024 |
| 096 | 0.15113 | 0.13076 | 0.27528 | 0.23579 | 0.63712 | 0.44440 |
| 120 | 0.15876 | 0.13399 | 0.28667 | 0.24234 | 0.71456 | 0.46962 |

Time step: old scheme: 30 minutes; new scheme: 90 minutes
CPU time: old scheme: 9.1 minutes; new scheme: 6.3 minutes

When vapor equation is solved by using this scheme, with the increasing of time step, however, the simulation of precipitation is affected greatly. The amount and area of precipitation decrease. In the following section, square-conservation semi-Lagrangian method is used to improve the computation of vapor equation.

## IV. COMPUTATION OF VAPOR EQUATION

Consider the following vapor equation (forcing term is neglected):

$$\frac{\partial q}{\partial t} = -\vec{V} \cdot \nabla q \ . \tag{9}$$

From continuity equation $\frac{\partial P_s}{\partial t} = -\nabla \cdot (\vec{V} P_s)$, vopor equation can be proved

$$\frac{dQ}{dt} = 0 \ , \tag{10}$$

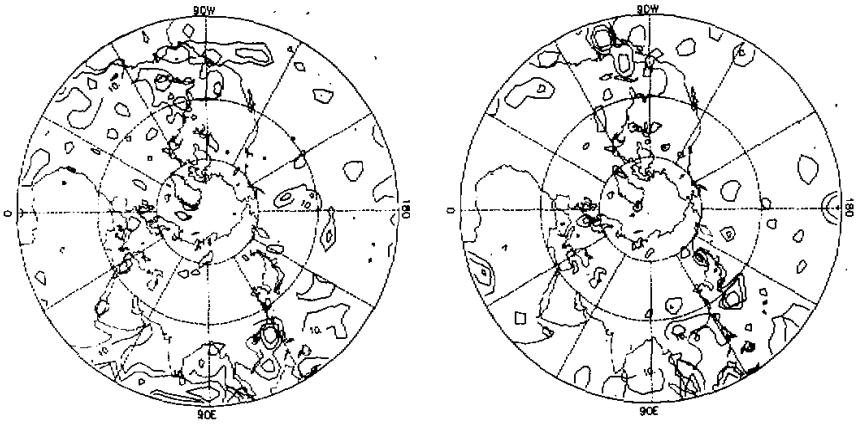
$$Q = \int_0^1 d\sigma \iint_\Omega \frac{1}{2} P_s q^2 ds \ , \tag{11}$$

where $\Omega$ is the spherical surface, $\vec{V} = (u,v,\dot{\sigma})$, $\nabla = \left(\frac{\partial}{a\cos\theta\partial\lambda}, \frac{\partial}{a\partial\theta}, \frac{\partial}{\partial\sigma}\right)$. So vapor equation (9) is of square conservative property. The following two conservative properties can also be proved:

$$\frac{d}{dt}\int_0^1 d\dot{\sigma} \iint_\Omega P_s q ds = 0 \ , \tag{12}$$

$$\frac{d}{dt}\int_0^1 d\dot{\sigma} \iint_\Omega \frac{1}{\sigma} P_s q^3 ds = 0 \ . \tag{13}$$

During the procedure of computing, approximate value $\tilde{q}_{i,jk}^{n+1}$ is calculated by linear




a. New scheme (multispectrum)

b. Old scheme (semi-implicit)

Fig. 1. Prediction of precipitation in the Northern Hemisphere (5th day).

interpolation formula and it can be proved that:

$$q_{i,j,k}^{n+1} - \tilde{q}_{i,j,k}^{n+1} = A\left(\frac{\partial^2 q}{\partial \lambda^2}\right)^{n+1} \Delta\lambda^2 + B\left(\frac{\partial^2 q}{\partial \theta^2}\right)^{n+1} \Delta\theta^2$$

$$+ C\left(\frac{\partial^2 q}{\partial \sigma^2}\right)^{n+1} \Delta\sigma^2 + O(\Delta\lambda^3 + \Delta\theta^3 + \Delta\sigma^3) ,$$

where $A$, $B$, $C$ are three undetermined coefficients. They can be determined by using the primary, quadratic and cubic conservative properties that $q_{i,j,k}^{n+1}$ satisfies. Because interpolation formula is used and not any extrapolation is involved, negative vapor can be avoided. The resulted $q_{i,j,k}^{n+1}$ not only keeps the primary, quadratic and cubic conservative properties but also is permanently positive. In practical computation because vapor equation consists of forcing term, the conservative part is first computed by using splitting method and then forcing term is involved.

Employing the observational data on June 1, 1992 as the initial field, the result of 5-day prediction is shown in Fig. 1. According to Fig. 1, the amount of simulated precipitation with the square-conservation semi-Lagrangian scheme is larger than that with the multispectrum scheme. Moreover, the amount of simulated precipitation with the square-conservation semi-Lagrangian method is the same order as that with the original semi-implicit scheme.

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