Frontogenesis, Evolution and the Time Scale of Front Formation

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ABSTRACT

Observational study shows that, in some cases, the frontal structure displays the features of gravitative flows. It seems that the formation of discontinuity is an important problem in the study of the frontogenesis which is usually defined as an increasing of the scalar gradient. In this paper, the characteristic features of air flow with initial imbalance between the wind and the density fields are studied. Much attention is paid on the condition for the formation of discontinuity and its time scale. It is found that the initial distribution of density plays an important role in the formation of the discontinuity which happens in short time duration.

Key words: Front, Frontogenesis, Evolution, Time scale

I. INTRODUCTION

Due to the enrichment of observational study on the front structure and the progress of modern techniques, the conception of front is suffering changes. In early stages, front was considered as a discontinuity surface. Later, this view point was replaced by a dramatic change zone in physical features. Recently, some scientists have pointed out that the front structure is of some features of gravitative flow. This means that the conception of front is now turning back to the discontinuity surface from the zone. Sharpiro and Keyser (1990) have ever made an excellent review on this point.

The change of conception will cause the change of the criterion on the frontogenesis. For example, according to the traditional point of view, the condition for frontogenesis is expressed as:

$$\frac{d}{dt}(|\nabla\theta| > 0 \ ,$$

where $\nabla\theta$ is the gradient of potentional temperature. However, this condition does not guarantee the occurrence of discontinuity which is an important phenomenon of the frontogenesis. So it seems not to be apt to be used as a criterion on the formation of discontinuity. Wu and Blumen (1995) and Blumen and Wu (1995) have ever discussed the occurrence of discontinuity or collapse of isopycnals in theoretical study with the potential vorticity conservation principle and geostrophic adjustment. Actually, Hoskins (1975) discussed the problem as early as in 1970's.

BW (1995) and Ou (1984) pointed out that the geostrophic adjustment is one possible process of frontogenesis. The geostrophic imbalance initial density and velocity fields will adjust to be geostrophic flows, and the final state can be the pattern of frontal collapse. With the initial imbalance fields we can determine the final state. However, the evolution of this process is not clear in BW (1995) and Ou (1984).

Simpson and Linden (1989) and Kay (1992) have done some research on frontogenesis in the non-rotating fluid and obtained the time scale of the frontogenesis and condition for the frontogenesis. Due to the ignorance of the rotation of the fluid, there is no discontinuity happened in their work, and their results are valid just for the small scale motion.

In this paper, the frontogenesis with imbalance initial fields is studied, the time dependent solutions, which are not necessary to be geostrophic, are obtained through the method proposed by Simpson and Linden (1984) and Kay (1992). The time scale for the formation of front is discussed in details with a simple 2-D atmosphere model.

II. BASIC EQUATIONS AND SOLUTIONS

The basic equations in (x, z) plane, with Boussinesq and hydrostatic approximations, are:

$$\frac{du}{dt} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} , \tag{1}$$

$$\frac{dv}{dt} + fu = 0 , \tag{2}$$

$$\frac{\partial p}{\partial z} = -\rho g , \tag{3}$$

$$\frac{d\rho}{dt} = 0 , \tag{4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 , \tag{5}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + w\frac{\partial}{\partial z} .$$

The notations of the variable are the same as used in meteorological paper. Eliminating p from (1) and (3) with Eq.(5), horizontal vorticity equation is obtained

$$\frac{d}{dt}\left(\frac{\partial u}{\partial z}\right) = \frac{g}{\rho}\frac{\partial \rho}{\partial x} + f\frac{\partial v}{\partial z} . \tag{6}$$

Eqs. (2), (4), (5) and (6) form a complete set of equations on u, v, w and ρ. Introducing following scales:

$$x = \frac{\sqrt{g^* H}}{f}x', \quad z = Hz', \quad t = \frac{1}{f}t' ,$$

$$(u, v) = \sqrt{g^* H}(u', v'), \quad w = fHw' , \tag{7}$$

$$\rho = \Delta\rho\rho', \quad g^* = g\Delta\rho / \rho_0 ,$$

then the governing equations can be rewritten as non-dimensional form:

$$\frac{d}{dt}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial \rho}{\partial x} + \frac{\partial v}{\partial z} , \tag{8}$$

$$\frac{dv}{dt}+u=0\ , \tag{9}$$

$$\frac{d\rho}{dt}=0\ , \tag{10}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0\ . \tag{11}$$

Eq.(9) can be rewritten as

$$\frac{d}{dt}(x+v)=0\ . \tag{12}$$

This means that $(x+v)$ is a conservative quantity. Now, we introduce momentum coordinates (X, Z, T). The relations between it and physical coordinates are:

$$X=x+v,\quad Z=z,\quad T=t. \tag{13}$$

Then, the governing equations under the momentum coordinates have the form as follows:

$$\left(1-\frac{\partial v}{\partial X}\right)\left(\frac{\partial}{\partial T}+w\frac{\partial}{\partial Z}\right)\Omega=\frac{\partial\rho}{\partial X}+\frac{\partial v}{\partial Z}\ , \tag{14}$$

$$\left(\frac{\partial}{\partial T}+w\frac{\partial}{\partial Z}\right)v+u=0\ , \tag{15}$$

$$\left(\frac{\partial}{\partial T}+w\frac{\partial}{\partial Z}\right)\rho=0\ , \tag{16}$$

$$\frac{\partial u}{\partial X}+\frac{\partial v}{\partial Z}\frac{\partial w}{\partial X}+\left(1-\frac{\partial v}{\partial X}\right)\frac{\partial w}{\partial Z}=0\ , \tag{17}$$

where

$$\Omega=\frac{\partial u}{\partial z}=\frac{\partial u}{\partial Z}+\frac{\partial u}{\partial X}\frac{\partial v}{\partial Z}\Big/\left(1-\frac{\partial v}{\partial X}\right)\ . \tag{18}$$

(See Appendix for the detailed deduction of Eqs.(14)–(18)).

The approximative solutions can be determined by expanding the variables in the series of T, that is:

$$r=r_0+r_1T+r_2T^2+\cdots, \tag{19}$$

where r stands for u, v, w, ρ, Ω. This expansion means that the variables are continuous in (X, Z, T) coordinates. However, this does not guarantee that they are continuous in physical coordinates. It was discussed in details in Hoskins (1975). For example, when

$$\frac{\partial v}{\partial X}-1=0\ , \tag{20}$$

the transformation between (x, z, t) and (X, Z, T) is terminated as pointed out in Hoskins (1975). In other words, in such situations, the discontinuity in (x, z, t) will take place. This is used as a criterion for the frontogenesis in following. If the Coriolis effect is ignored as Simpson and Linden (1989) and Kay (1992), then (X, Z, T) space and (x, z, t) space are identical. When the flow in (X, Z, T) space is continuous, then, it is continuous in physical space either.

Substituting (19) into Eqs.(14–18) and rearranging them as power set of T. Then, we have

$$T^0:\begin{cases}(1-\dfrac{\partial v_0}{\partial X})\left(\Omega_1+w_0\dfrac{\partial \Omega_0}{\partial Z}\right)=\dfrac{\partial \rho_0}{\partial X}+\dfrac{\partial v_0}{\partial Z}\\ v_1+u_0+w_0\dfrac{\partial v_0}{\partial Z}=0\\ \rho_1+w_0\dfrac{\partial \rho_0}{\partial Z}=0\\ \dfrac{\partial u_0}{\partial X}+\dfrac{\partial v_0}{\partial Z}\dfrac{\partial w_0}{\partial X}+\left(1-\dfrac{\partial v_0}{\partial X}\right)\dfrac{\partial w_0}{\partial Z}=0\\ (1-\dfrac{\partial v_0}{\partial X})\Omega_0=\left(1-\dfrac{\partial v_0}{\partial X}\right)\dfrac{\partial u_0}{\partial Z}+\dfrac{\partial u_0}{\partial X}\dfrac{\partial v_0}{\partial Z}\end{cases}\tag{21}$$

$$T^1:\begin{cases}2(1-\dfrac{\partial v_0}{\partial X})\Omega_2=\dfrac{\partial \rho_1}{\partial X}+\dfrac{\partial v_1}{\partial Z}+\dfrac{\partial v_1}{\partial X}\Omega_1+\dfrac{\partial v_1}{\partial X}w_0\dfrac{\partial \Omega_0}{\partial Z}-w_1\dfrac{\partial \Omega_0}{\partial Z}\\ \quad -w_0\dfrac{\partial \Omega_1}{\partial Z}+\dfrac{\partial v_0}{\partial X}w_1\dfrac{\partial \Omega_0}{\partial Z}+\dfrac{\partial v_0}{\partial X}w_0\dfrac{\partial \Omega_1}{\partial Z}\\ 2v_2+u_1+w_0\dfrac{\partial v_1}{\partial Z}+w_1\dfrac{\partial v_0}{\partial Z}=0\\ 2\rho_2+w_0\dfrac{\partial \rho_1}{\partial Z}+w_1\dfrac{\partial \rho_0}{\partial Z}=0\\ \dfrac{\partial u_1}{\partial X}+\left(\dfrac{\partial v_0}{\partial Z}\dfrac{\partial w_1}{\partial X}+\dfrac{\partial v_1}{\partial Z}\dfrac{\partial w_0}{\partial X}\right)+\left(1-\dfrac{\partial v_0}{\partial X}\right)\dfrac{\partial w_1}{\partial Z}-\dfrac{\partial v_1}{\partial X}\dfrac{\partial w_0}{\partial Z}=0\\ \left(1-\dfrac{\partial v_0}{\partial X}\right)\Omega_1=\dfrac{\partial v_1}{\partial X}\Omega_0+\left(1-\dfrac{\partial v_0}{\partial X}\right)\dfrac{\partial u_1}{\partial Z}-\dfrac{\partial u_0}{\partial Z}\dfrac{\partial v_1}{\partial X}+\left(\dfrac{\partial u_1}{\partial X}\dfrac{\partial v_0}{\partial Z}+\dfrac{\partial u_0}{\partial X}\dfrac{\partial v_1}{\partial Z}\right)\end{cases}\tag{22}$$

For given u_0, v_0, ρ_0, w_0 and Ω_0, we can subsequently find the quantities with subscripts 1,2 and so on. As a result, the series of solutions are obtained for the governing equations.

From (21), it is plausible that the initial velocity distribution cannot be arbitrary. One possible relation is $u_0, v_0, w_0=0$. This will be used in the following study.

The initial variables considered in this paper are expressed as:

$$u_0=0,\quad v_0=0,\quad w_0=0\ .\tag{23}$$

$$\rho_0=\rho_0(X,Z)\ .$$

Then Eq.(21) reduces as

$$\begin{cases}\Omega_1=\dfrac{\partial \rho_0}{\partial X} & (a)\\ v_1=0 & (b)\\ \rho_1=0 & (c)\end{cases}\tag{24}$$

(22) becomes:

$$
\begin{cases}
\Omega_2 = 0 & (a) \\
v_2 = -\frac{1}{2}u_1 & (b) \\
\rho_2 = -\frac{1}{2}w_1\frac{\partial \rho_0}{\partial Z} & (c) \\
\frac{\partial u_1}{\partial X} + \frac{\partial w_1}{\partial Z} = 0 & (d) \\
\Omega_1 = \frac{\partial u_1}{\partial Z} & (e)
\end{cases}
\tag{25}
$$

Assuming that the lower boundary and upper boundary are rigid, then from (25.d), we can find

$$\int_0^1 u_1\, dz = \text{cosst} \ .$$

With this constraint, from (24.a) and (25.e), u_1 and w_1 are obtained

$$
\begin{cases}
u_1 = \int_0^Z \frac{\partial \rho_0}{\partial X} dZ - \int_0^1 \int_0^Z \frac{\partial \rho_0}{\partial X} dZdZ \\
w_1 = -\int_0^Z \int_0^Z \frac{\partial^2 \rho}{\partial X^2} dZdZ + Z\int_0^1 \int_0^Z \frac{\partial^2 \rho}{\partial X^2} dZdZ \ ,
\end{cases}
\tag{26}
$$

and then, we can determine u_2, v_2, w_2 and ρ_2 in sequence as:

$$
\begin{cases}
v_2 = -\frac{1}{2}u_1 = -\frac{1}{2}(\int_0^Z \frac{\partial \rho_0}{\partial X} dZ - \int_0^1 \int_0^Z \frac{\partial \rho_0}{\partial X} dZdZ) \\
u_2 = 0 \\
w_2 = 0 \\
\rho_2 = \frac{1}{2}\frac{\partial \rho_0}{\partial Z}(\int_0^Z \int_0^Z \frac{\partial^2 \rho}{\partial X^2} dZdZ - Z\int_0^1 \int_0^Z \frac{\partial^2 \rho}{\partial X^2} dZdZ) \ .
\end{cases}
\tag{27}
$$

Finally, the approximative solutions of (14–18) are obtained by substituting (26), (27) into (19). For example, v component can be expressed as

$$v = v_2 t^2 = -\frac{1}{2}(\int_0^Z \frac{\partial \rho_0}{\partial X} dZ - \int_0^1 \int_0^Z \frac{\partial \rho_0}{\partial X} dZdZ)t^2 \ . \tag{28}$$

As it has been stated above, the transformation between momentum and physical coordinates is terminated and the discontinuity is formed when (20) is satisfied. Under this condition, using (28), we have

$$\frac{\partial v_2}{\partial X} t^2 = 1 \ .$$

Accordingly, the time scale of the formation of discontinuity can be determined approximately, that is:

$$t_c = \min\left(\frac{1}{\sqrt{\frac{\partial v_2}{\partial X}}}\right) = \min\left(\sqrt{\frac{2}{-\int_0^Z \frac{\partial^2 \rho_0}{\partial X^2} dZ + \int_0^1 \int_0^Z \frac{\partial^2 \rho_0}{\partial X^2} dZ dZ}}\right) . \tag{29}$$

For the surface front which is usually considered, the time scale is:

$$t_c = \min\left(\frac{1}{\sqrt{\frac{\partial v_2}{\partial X}}}\right) = \min\left(\sqrt{\frac{2}{\int_0^1 \int_0^Z \frac{\partial^2 \rho_0}{\partial X^2} dZ dZ}}\right) . \tag{30}$$

Obviously, the gradient of the initial density plays an important role in the formation of the discontinuity, i.e., the distribution of density should be quadratic at least. Otherwise, t_c will tend to infinite and no front can be formed. This agrees with the conclusion of Simpson and Linden (1989). Since the series expansion of (19) is convergent only under the condition that t_c is far less than one, so another requirement of the distribution of density for the formation of the discontinuity is

$$\max\left(\int_0^1 \int_0^Z \frac{\partial^2 \rho_0}{\partial X^2} dZ dZ\right) >> 2 . \tag{31}$$

This ensures the discontinuity can be formed within a limited time duration.

With the results obtained above, two simple cases will be studied in the next section.

III. TWO SIMPLE CASES

In (X, Z, T) coordinates, the potential vorticity reads as

$$P.V. = \frac{\partial \rho}{\partial Z} \frac{1}{1 - \frac{\partial v}{\partial X}} .$$

The derivation in details can be found in BW. The potential vorticity conservation principle can be expressed as

$$\frac{\partial \rho}{\partial Z} \frac{1}{1 - \frac{\partial v}{\partial X}} = P.V._0 = \text{initial potential vorticity.} \tag{32}$$

If the initial velocity $v_0 = 0$ then q_0 depends on the distribution of ρ. For instance, if ρ_0 is only the function of X, then q_0 equals zero which is called as zero potential vorticity flow in BW and WB. Alternatively, if ρ_0 is the function of X and Z, but it is linearly dependent on Z, then $\frac{\partial \rho}{\partial Z}$ = const. This case is called uniform potential vortcity flow. These two cases are special simplified cases to simulate the atmospheric conditions. They have been used in many studies. Since the uniform P.V. flow is more interesting than zero P.V. flow in the real atmosphere, in this paper, we will just discuss the former, and then study more complicated case, non-uniform potential vorticity flow.

Case 1. Uniform potential vorticity flow

The initial conditions are specified as

$$u_0 = 0, \quad v_0 = 0, \quad w_0 = 0 \ ,$$
$$\rho_0 = \rho_0^* (x) - z \ . \tag{33}$$

Since X is conservative, i.e.

$$X = X_0 = x_0 + v_0 \ ,$$

where x_0 is the initial position of a parcel, v_0 is the velocity of the parcel. If the initial condition is motionless as shown in (23), then

$$X = x_0 \ .$$

The initial distribution of the density is

$$\rho_0 = \rho_0^* (x_0) - z = \rho_0^* (X) - Z \ . \tag{34}$$

Accordingly, the solution of (14–18) with this initial condition is:

$$\begin{cases} u_1 = \dfrac{\partial \rho_0^*}{\partial X}(Z - \dfrac{1}{2}) \\ w_1 = -\dfrac{1}{2}\dfrac{\partial^2 \rho_0^*}{\partial X^2}(Z^2 - Z) \\ v_1 = 0 \\ \rho_1 = 0 \ . \end{cases} \tag{35}$$

and finally, we can determine u_2, v_2, w_2 and ρ_2 in sequence as:

$$\begin{cases} v_2 = -\dfrac{1}{2}u_1 = -\dfrac{1}{2}\dfrac{\partial \rho_0^*}{\partial X}(Z - \dfrac{1}{2}) \\ u_2 = 0 \\ w_2 = 0 \\ \rho_2 = -\dfrac{1}{4}\dfrac{\partial^2 \rho_0^*}{\partial X^2}(Z^2 - Z) \ . \end{cases} \tag{36}$$

If the form of ρ_0^* is given then the velocity field and density field in (X, Z, T) coordinates can be found. Using the definition of X, the solution can be recovered to the physical coordinates (x, z, t).

According to (30), the time scale for the formation of discontinuity for this case is

$$t_c = \min \left(\sqrt{\frac{4}{\dfrac{\partial^2 \rho_0}{\partial X^2}}} \right) \ . \tag{37}$$

The another requirement of the distribution of the density, according to (31), is expressed as:

$$\max\left(\frac{\partial^2 \rho_0}{\partial X^2}\right) > > 4 \ , \qquad (38)$$

in nondimensional form. These two conditions ensure the formation of the surface front within a short period of time.

Next, if ρ_0^* is taken as the form

$$\rho_0^*(X) = \alpha Erf(bX) \ ,$$

where a and b are constants. Then, the distributions of ρ and $\vec{v}$ can be found. Fig.1 (a) shows the initial distribution of density. Fig.2 shows the evolution of the velocity fields and density field. At $t = 0.5$, the isotaches of v and isostathes converge near $(x, z) = (0.2, 0)$. But no discontinuity is formed. But at $t = t_c$, which is 0.68, calculated by means of Eq.(37), the discontinuity is obvious in the fields of v and ρ. In the density field, isostathes of $\rho = -0.2$ and $\rho = -0.4$ collapse each other at $(x, z) = (0.23, 0)$, which means that at the point (0.23, 0) the gradients of ρ and v are infinite. This is an important feature of the discontinuity or frontogenesis. Fig.2 (c) is the distribution of vertical velocity at $t = t_c$. It is obvious that there is a strong upward motion ($w_{max} = 0.7$ m / s) above the surface front and a strong downward motion behind the surface front for compensation. Fig.2 (d) shows the distribution of the horizontal kinetic energy at $t = t_c$. Near the surface and the upper front, there are two high horizontal velocity regions which are something like the low level jets in the real atmosphere. From this, a conclusion can be drawn that the geostrophic adjustment is also a possible mechanism for the formation of low level jets.

Case 2. The simple non-uniform potential vorticity flow

Assuming that the initial density has the form

$$\rho_0(X,Z) = \rho_0^*(X)(1 - Z) - Z \ , \qquad (39)$$

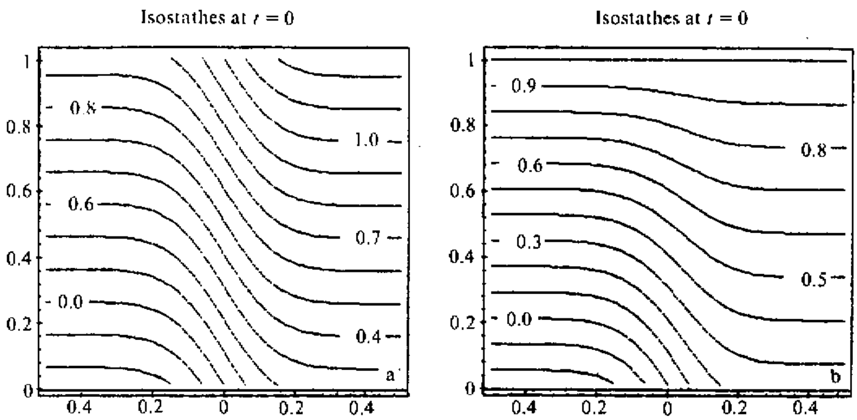


Fig. 1. The distribution of initial density of uniform P. V. and non-uniform P. V. flow. a: Uniform P. V. flow, b: Non-uniform P. V. flow

as a result, the potential vorticity is obtained as

$$P.V.=P.V._0=-\rho_0^*(X)-1\ ,$$

from Eq.(32), which means that the distribution of P.V. is non-uniform. The initial distribution of density is shown in Fig.1 (b).

Then the u, v, w and ρ are obtained:

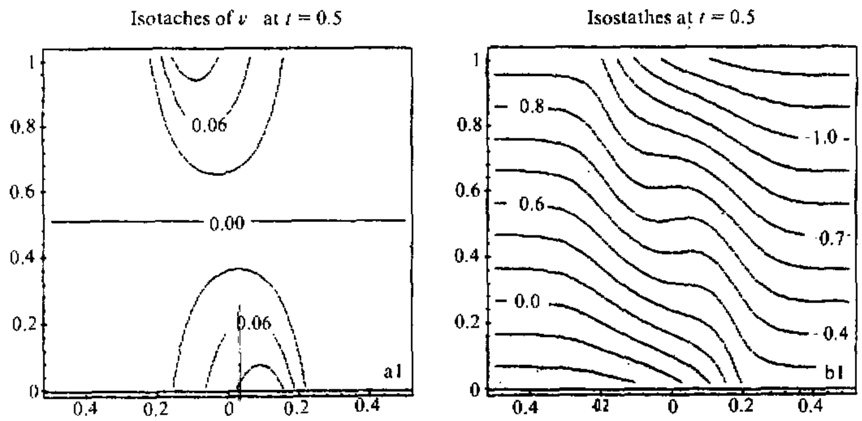
$$u=\frac{\partial\rho_0^*}{\partial X}(Z-\frac{1}{2}Z^2-\frac{1}{3})t\ ,$$

$$v=-\frac{1}{2}\frac{\partial\rho_0^*}{\partial X}(Z-\frac{1}{2}Z^2-\frac{1}{3})t^2\ ,$$

$$w=-\frac{\partial^2\rho_0^*}{\partial X^2}(\frac{1}{2}Z^2-\frac{1}{6}Z^3-\frac{1}{3}Z)t\ ,$$

$$\rho=\rho_0^*(X)(1-Z)-Z-\frac{1}{2}\frac{\partial^2\rho_0^*}{\partial X^2}\left(\frac{\partial\rho_0^*}{\partial X}+1\right)\left(\frac{1}{2}Z^2-\frac{1}{6}Z^3-\frac{1}{3}Z\right)t^2\ .$$

Fig. 3 shows the distribution of velocity and density field for this case at $t=t_c=0.83$, which displays the main features of front as Fig.2. Near the surface, the isotaches of v collapse at $x=0.23$, and isostathes of $\rho=-0.2$ and $\rho=-0.4$ also collapse each other. Above the surface front, the vertical velocity is very strong. And near the surface there exists a region of high horizontal kinetic energy which is something like the front jets as showed in Fig.2(d). However, there is a prominent difference between Fig.2 and Fig.3, which is, in Fig.2, there is no front formed at the upper layer. Neither the velocity field nor the density field shows the discontinuity. This phenomenon can be explained as, at the surface, the gradient of



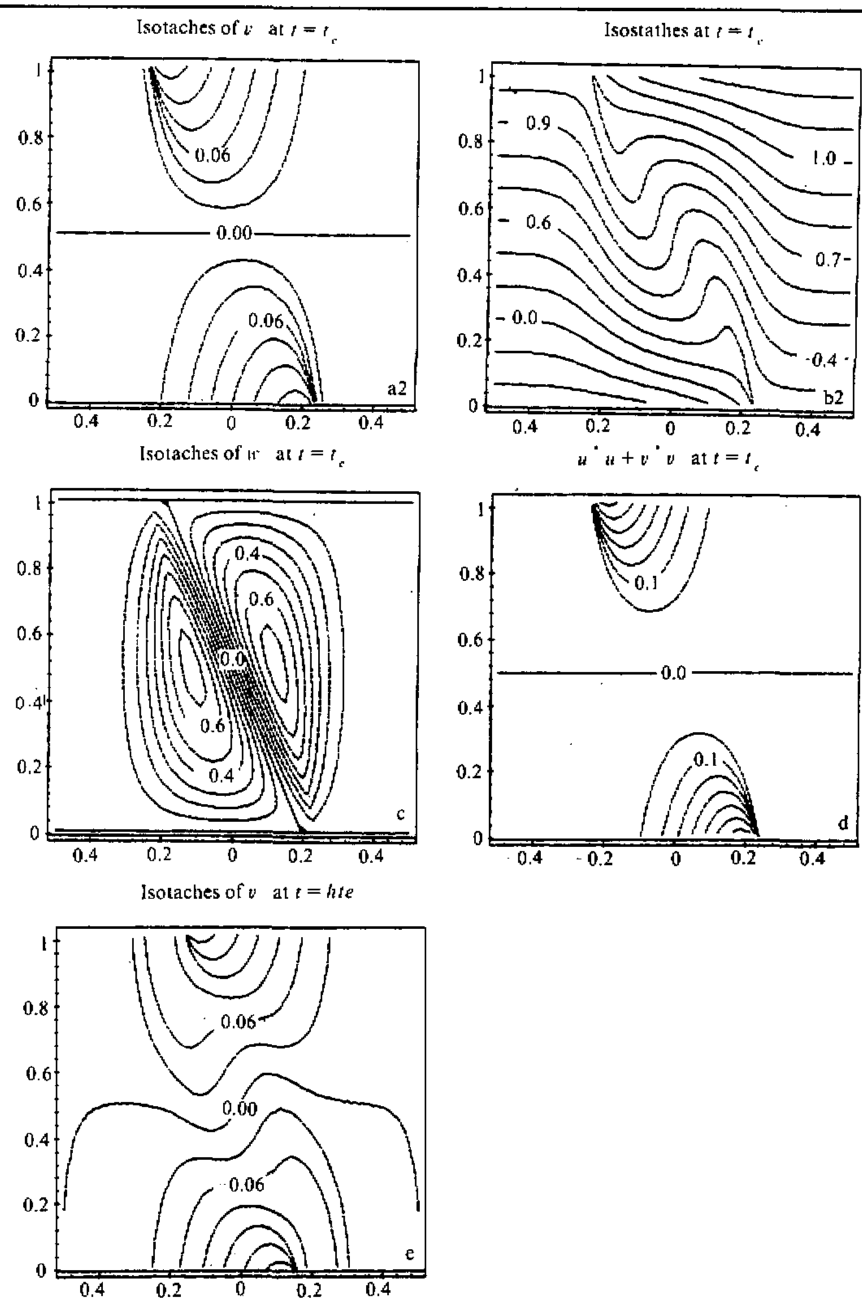


Fig. 2. The case of uniform P. V. flow. a1: Isotaches of v at $t = 0.5$, a2: Isotaches at $t = t_c = 0.68$, b1: Isostathes at $t = 0.5$, b2: Isostathes at $t = t_c$, c: Isotaches of w at $t = t_c$, d: $u^2 + v^2$ at $t = t_c$, e: Isotaches of v at $t = t_c = 0.73$ (correction of high order term is contained).

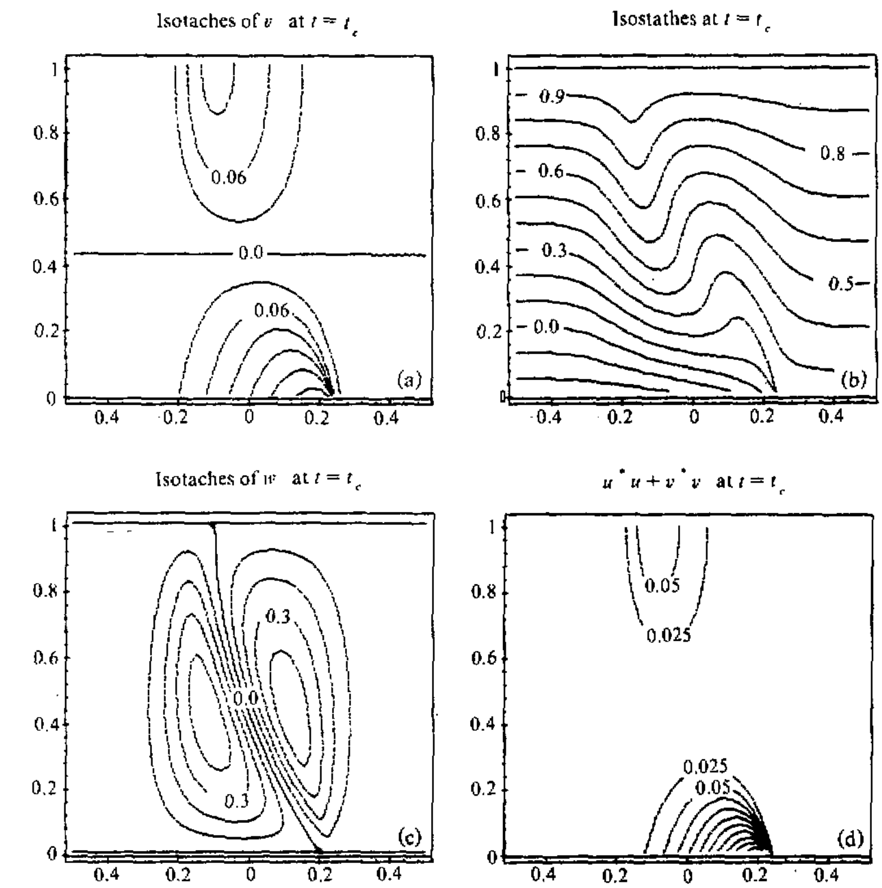


Fig.3. The case of non-uniform P. V. flow. a: Isotaches of v at $t = t_c = 0.83$, b: Isostathes at $t = t_c$, c: Isotaches of w at $t = t_c$, d: $u^2 + v^2$ at $t = t_c$

the initial density $\frac{\partial \rho_0}{\partial X}$ being large, so it produces large pressure force. Since the gradient of the initial density is quadratic at least, the pressure force is not equal everywhere, strong convergent motion is produced in certain region, and finally causes the formation of the discontinuity. While, at the top boundary, due to the small gradient of the initial density, the vertical transportation of mass, caused by the horizontal convergent motion near the surface, is not large enough to increase the gradient of density prominently to lead to the formation of the upper front at the same time as the surface front. This proves the conditions for the formation of discontinuity derived above (i.e.(30) and (31)).

IV. THE HIGH ORDER CORRECTION

The two cases above are both considered by using one order approximation. In this section, we will explore the corrections to them of high order terms. For simplicity, just the case of uniform potential vorticity is considered.

Substituting (19) into (14–18) and using the initial distribution of density for the uniform case, high order terms of u and v are obtained as:

$$u_3=\frac{1}{6}\frac{\partial\rho_0^*}{\partial X}\frac{\partial^2\rho_0^*}{\partial X^2}(Z^2-Z)-\frac{1}{36}\frac{\partial^3\rho_0^*}{\partial X^3}(Z^3-\frac{3}{2}Z^2)-\frac{1}{6}\frac{\partial\rho_0^*}{\partial X}Z$$

$$+\frac{1}{144}\int_{-\frac{1}{2}}^{X}10\frac{\partial\rho_0^*}{\partial X}\frac{\partial^3\rho_0^*}{\partial X^3}+12\frac{\partial^2\rho_0^*}{\partial X^2}+10\left(\frac{\partial^2\rho_0^*}{\partial X^2}\right)^2-\frac{\partial^4\rho_0^*}{\partial X^4}dX \tag{41}$$

$$v_4=-\frac{1}{4}u_3-\frac{1}{16}\frac{\partial\rho_0^*}{\partial X}\frac{\partial^2\rho_0^*}{\partial X^2}(Z^2-Z)\ .$$

Fig. 2(e) shows the distribution of v at $t=t_c=0.73$ when the high order term $O(t^4)$ is contained. It is obvious that the main feature of the frontogenesis is the same as the low order approximation shown in Fig.2(a2) except there is a little correction to the low order solution. For example, the position of the surface front is changed a little and the time scale for the formation of discontinuity turns longer $t_c=0.73$, while $t_c=0.68$ for $O(t^2)$. Despite these corrections, the low order approximation does express the main characteristic of the process of the frontogenesis, i.e. the formation of the discontinuity on the whole.

V. CONCLUDING REMARKS

In this work, simple model and simple method are used to discuss the formation of frontogenesis and the time scale for frontogenesis. The main results are:

1) Without the large scale initial velocity fields, the initial density field also can form the surface front under the condition that the distribution of density is, at least, quadratic and $\max(\int_0^1\int_0^Z\frac{\partial^2\rho_0}{\partial X^2}dZdZ)$ is far larger than 2.

2) The gradient of density plays an important role in the frontogenesis.

3) The time scale for frontogenesis is related to $\frac{\partial^2\rho_0}{\partial X^2}$.

In this simple model, the effects of dissipation and orography are not contained. In fact, during the frontogenesis, the effect of dissipation will become important due to the increase of the gradient of variables. On the other hand, the orography has obvious effect on the real atmosphere. So it is necessary to consider the effects of dissipation and orography in the problem of frontogenesis.

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Appendix

The deduction of Eqs.(14)–(18)

According to the relations between the physical space and the momentum space which are expressed in (13), we have

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial T}+\frac{\partial v}{\partial t}\frac{\partial}{\partial X} \tag{A1.a}$$

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial X}+\frac{\partial v}{\partial x}\frac{\partial}{\partial X} \tag{A1.b}$$

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial Z}+\frac{\partial v}{\partial z}\frac{\partial}{\partial X}\ . \tag{A1.c}$$

With these, $\frac{d}{dt}$ can be written as

$$\begin{aligned}\frac{d}{dt}&=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+w\frac{\partial}{\partial z}\\&=\frac{\partial}{\partial T}+\frac{\partial v}{\partial t}\frac{\partial}{\partial X}+u\left(\frac{\partial}{\partial X}+\frac{\partial v}{\partial x}\frac{\partial}{\partial X}\right)+w\left(\frac{\partial}{\partial Z}+\frac{\partial v}{\partial z}\frac{\partial}{\partial X}\right)\ ,\end{aligned}$$

which, after some modifications, has the following form

$$\frac{d}{dt}=\frac{\partial}{\partial T}+w\frac{\partial}{\partial Z}+u\frac{\partial}{\partial X}+\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+w\frac{\partial v}{\partial z}\right)\frac{\partial}{\partial X}\ . \tag{A2}$$

According to Eq.(9), (A2) is simplified as

$$\frac{d}{dt}=\frac{\partial}{\partial T}+w\frac{\partial}{\partial Z}\ . \tag{A3}$$

Using the relations expressed in (A1.b,c), $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial z}$ can be expressed as

$$\begin{aligned}\frac{\partial v}{\partial x}&=\frac{\partial v}{\partial X}\left(1-\frac{\partial v}{\partial X}\right)^{-1}\ ,\\\frac{\partial v}{\partial z}&=\frac{\partial v}{\partial Z}\left(1-\frac{\partial v}{\partial X}\right)^{-1}\ .\end{aligned} \tag{A4}$$

Then, (A1.b,c) have the forms as follows:

$$\begin{aligned}\frac{\partial}{\partial x}&=\left(1-\frac{\partial v}{\partial X}\right)^{-1}\frac{\partial}{\partial X}\ ,\\\frac{\partial}{\partial z}&=\frac{\partial}{\partial Z}+\frac{\partial v}{\partial Z}\left(1-\frac{\partial v}{\partial X}\right)^{-1}\frac{\partial}{\partial X}\ .\end{aligned} \tag{A5}$$

Substituting (A3), (A5) into (8)–(11), Eqs.(14)–(18) are obtained after some manipulating.

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