On the Forcing of the Radial–vertical Circulation within Cyclones—Part 1: Concepts and Equations

D. R. Johnson and *Zhuojian Yuan*

Space Science and Engineering Center, University of Wisconsin–Madison, Madison, WI 53706, USA

Received August 11, 1997; revised November 5, 1997

ABSTRACT

Following the theoretical result of Eliassen, the Sawyer–Eliassen equation for frontal circulations and the equation for forcing the meridional circulation within a circumpolar vortex are extended in isentropic coordinates to describe the forcing of the azimuthally averaged mass–weighted radial–vertical circulation within translating extratropical and tropical cyclones. Several physical processes which are not evident in studies employing isobaric coordinates are isolated in this isentropic study. These processes include the effects of pressure torque, inertial torque and storm translation that are associated with the asymmetric structure in isentropic coordinates. This isentropic study also includes the effects of eddy angular momentum transport, diabatic heating and frictional torque that are common in both isentropic and isobaric studies. All of the processes are modulated by static, inertial and baroclinic stabilities.

Consistent with the theoretical result of Eliassen, the numerical solution from this isentropic study shows that the roles of torque, diabatic heating and hydrodynamic stability in forcing the radial–vertical circulation within stable vortices are that 1) positive (negative) torque which results in the counterclockwise (clockwise) rotation of vortices also forces the outflow (inflow) branch of the radial–vertical circulation, 2) diabatic heating (cooling) forces the ascent (descent) branch of the radial–vertical circulation and 3) for given forcing, the weaker hydrodynamic stability results in a stronger radial–vertical circulation. It is the net inflow or convergence (net outflow or divergence), vertical motions and the associated redistribution of properties that favor the evolution of vortices with colorful weather events.

Numerical solutions of this isentropic study are given in companion articles. The relatively important contribution of various physical processes to the forcing of the azimuthally–averaged mass–weighted radial–vertical circulation within different translating cyclones and in their different stages of development will be investigated.

Key words: The forcing of the radial–vertical circulation, Quasi–Lagrangian diagnostics of cyclones, Balanced equation

I. INTRODUCTION

One property that has been used to measure the intensity of cyclones is the absolute angular momentum about the local vertical axis of the cyclone (Johnson and Downey, 1975a). Several absolute angular momentum and mass budget studies of extratropical cyclones with the use of quasi–Lagrangian perspective have been carried out by Johnson and Downey (1976), Hale (1983) and Johnson and Hill (1987). Their studies indicate that a net outward transport of mass and a net inward transport of absolute angular momentum are required in the development and maintenance of cyclones. These mass and angular momentum transports are accomplished through a systematic radial–vertical mass circulation within cyclones (Johnson and Downey, 1975a). Analysis and numerical simulation of this radial–vertical circulation will aid in understanding the dynamical forcing processes of cyclogenesis and development.

A quasi-static theory of meridional circulation within a stable and symmetric circumpolar vortex has been developed by Eliassen (1951). In his theory heating and frictional torque combine to force the meridional mass circulation within the vortex. In a hydrodynamically stable vortex, the role of the torques is to force the horizontal branches of the mass circulation through the surfaces of angular momentum, and the role of diabatic processes is to force the vertical branches through isentropic surfaces. In this manner the mass circulation is maintained in the presence of hydrodynamic stability and serves to supply the angular momentum and energy needed to sustain the vortex.

With an extension of Eliassen's work, Kuo (1956) studied the isobaric zonally averaged, hemispheric meridional circulation response to sources and sinks of heat and angular momentum. He found that the effects of the horizontal convergence of eddy angular momentum transport (eddy mode), heat transport, diabatic heating and frictional dissipation of zonal momentum tend to produce a three-cell meridional circulation in the troposphere with a middle-latitude indirect cell and tropical and polar direct cells.

Recognizing that a direct meridional circulation is forced in isentropic coordinates by large scale heat sources and sinks, Dutton (1976) introduced the concept of forced zonally averaged general circulation in isentropic coordinates. Using this concept and an approach similar to Eliassen's (1951), Gallimore and Johnson (1981) developed a diagnostic equation for the forced isentropic meridional circulation within an asymmetric circumpolar vortex. In their equation, frictional torque, pressure torque and eddy mode force the quasi-horizontal motion along isentropic surfaces. Heating and cooling force motions through isentropic surfaces. Their results show a direct meridional circulation in each hemisphere in response to the systematic meridional distribution of heating in tropical latitudes and cooling in polar latitudes.

Eliassen's balanced vortex theory has also been a basis for tropical vortex studies. Sundqvist (1970), Challa and Pfeffer (1980) and Pfeffer and Challa (1981) completed similar studies involving the numerical modeling of stationary tropical cyclones. Sundqvist included a diagnostic equation for the forcing of the radial-vertical circulation within symmetric tropical cyclones, and modeled the characteristics of tropical cyclone development successfully. By extending Sundqvist's work to asymmetric tropical cyclones and including the effect of eddy mode, Challa and Pfeffer improved the simulation of tropical cyclone development.

Pfeffer and Challa's result that upper tropospheric eddy angular momentum fluxes can result in the intensity change of tropical cyclones has been confirmed by the analyses of outflow layer winds and the radial convergence of eddy flux of tangential velocity in Hurricane Elena (Molinari and Vollaro, 1990). With the use of Eliassen's balanced vortex equation in storm-relative coordinates, Molinari and Vollaro (1990) investigated the interrelationship between the intensity change of Elena, the azimuthally averaged radial-vertical circulation within Elena, lateral and vertical eddy fluxes of absolute angular momentum and potential temperature. The balanced solutions showed that the contribution of eddy heat fluxes (equivalent to the pressure torque in isentropic coordinates) to the radial-vertical circulation was in the same manner as momentum fluxes near the core, but with smaller magnitude and areal converge. Their further work suggested that the isolation of physical processes involving eddy fluxes was meaningful only in the storm-relative framework provided by the moving coordinate (Molinari et al., 1993).

Eliassen's balanced vortex equation has been modified by Schubert et al. (1987) into an equation governing the inverse potential vorticity for tropical cyclones. The balanced tangential wind fields obtained from the inverse potential vorticity are similar in many ways to those observed within tropical cyclones. Numerical solutions of this equation demonstrate how latent heat release generates potential vorticity at low levels and destroys it at upper levels.

The objectives of this paper are twofold. The first is to apply the methods developed by Eliassen (1951) and Gallimore and Johnson (1981) to derive a diagnostic equation for the forcing of the azimuthally averaged mass–weighted radial–vertical circulation within stable, asymmetric and translating extratropical and tropical cyclones. The forcing terms in this equation include the pressure torque (representing the baroclinic structure of atmosphere and eddy heat flux), frictional torque, diabatic heating and eddy modes. In addition, the effect of asymmetric mass distribution, the effects of rotation of the earth, the acceleration of cyclone's movement and the tilting of local vertical axis of rotation with respect to the earth's rotation axis due to cyclone's movement over the spherical earth are included. The second objective is to apply the diagnostic equation, the theories developed by Eliassen (1951) and Gallimore and Johnson (1981) and results of mass and angular momentum budget studies of cyclones (Johnson, 1974; Johnson and Downey, 1975b, c; Katzfey, 1978; Schneider, 1986; Johnson and Hill, 1987) to the discussion of the role of individual physical process in forcing the radial–vertical circulation within cyclones.

II. THE BASIC CONCEPTS AND BASIC EQUATIONS

1. *The Angular Momentum and Quasi–Lagrangian Framework*

The constraints on mass and absolute angular momentum provide basic insight into the forcing processes during the evolution of cyclones. These constraints are imposed from the conditions 1) that mass is conserved and 2) that increasing absolute angular momentum about a local storm axis can only be realized by an inward transport of the property. The requirements of a net outward mass transport and a net inward transport of absolute angular momentum in a developing cyclone dictate that both inward and outward branches of a mass circulation exist. The inward branch must transport more absolute angular momentum into the cyclone than the outward branch removes, while more mass is removed by the outward branch than is imported by the inward branch. This implies that in a stratified atmosphere there may be systematic inflow within certain isentropic layers and systematic outflow in other layers (Johnson and Downey, 1975b).

The ability to resolve these processes in the westerly wind regimes of the extratropical latitudes and easterly wind regimes of tropical latitudes through diagnostics depends crucially on two subtle and important steps. The first step is to define a quasi–Lagrangian coordinate system which moves with the velocity of the storm in order to separate the horizontal advection $\vec{W} \cdot \nabla_\theta (f)$ or transport $\nabla_\theta \cdot [(f)\vec{W}]$ associated with wave translation from the relative advection $(\vec{U} - \vec{W}) \cdot \nabla_\theta (f)$ or transport $\nabla_\theta \cdot [(f)(\vec{U} - \vec{W})]$ associated with development, where f represents any arbitrary property. The second step is to define an absolute angular momentum

$$\vec{g}_a = (\vec{r} - \vec{r}_o) \times (\vec{U}_a - \vec{W}_{oa}), \tag{1}$$

which is consistent with the quasi–Lagrangian coordinate system. In (1), $\vec{r}_o$ is the radius vector to the storm center and $\vec{W}_{oa}$ is its absolute velocity (Fig. 1). The angular momentum

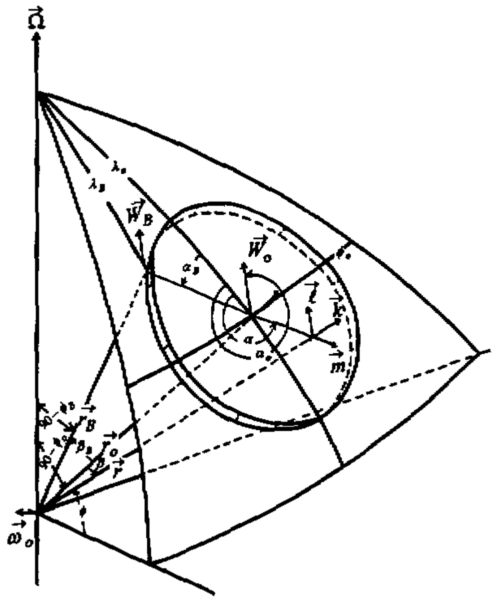

Fig.1. The storm spherical coordinate system (Johnson and Downey, 1975a).

defined by (1) is the moment of momentum about the center of the storm vortex, which increases during the development of cyclone and decreases during decay (Johnson and Downey, 1975b).

With regard to the relative effect of storm translation, its importance lies in the modes of transport within asymmetric vortices and the movement of the origin about which the moment of momentum exists. Since the storm vortex can develop only with net outward mass and net inward angular momentum transport relative to the storm center (Johnson and Downey, 1976), the isolation of the physical processes responsible for forcing the inward and outward transports will aid in understanding the evolution of cyclones.

2. *The Azimuthally Averaged Transport Equations*

In this isentropic study all the horizontal and time derivations are performed on isentropic surfaces, which implies that $\partial(\ )/\partial t, \partial(\ )/\partial\beta$, and $\partial(\ )/\partial\alpha$ in the following equations represent $\partial(\ )/\partial t_\theta, \partial(\ )/\partial\beta_\theta$ and $\partial(\ )/\partial\alpha_\theta$ respectively. The azimuthally averaged transport equations in isentropic hydrostatic quasi-Lagrangian spherical coordinates (Johnson and Downey, 1975a, b) include: the mass continuity equation

$$\frac{\delta}{\delta t}\left(\frac{\partial\bar{p}}{\partial\theta}\right)+\frac{1}{a\sin\beta}\frac{\partial}{\partial\beta}\left[\frac{\partial\bar{p}}{\partial\theta}(\hat{u}-\hat{w})_{\beta}\sin\beta\right]+\frac{\partial}{\partial\theta}\left(\frac{\partial\bar{p}}{\partial\theta}\hat{\dot{\theta}}\right)=0, \tag{2}$$

and the equation for the vertical component of storm absolute angular momentum g_{az}

$$\frac{\delta}{\delta t}(\frac{\partial \bar{p}}{\partial \theta}\hat{g}_{az}) + \frac{1}{a\sin\beta}\frac{\partial}{\partial \beta}[\frac{\partial \bar{p}}{\partial \theta}(\hat{u} - \hat{w})_\beta \hat{g}_{az}\sin\beta] + \frac{\partial}{\partial \theta}(\frac{\partial \bar{p}}{\partial \theta}\hat{g}_{az}\hat{\dot{\theta}}) = \frac{\partial \bar{p}}{\partial \theta}\hat{F}. \tag{3}$$

where w_β is the radial component of velocity of storm center,

$$\hat{F} = -<\frac{\partial \psi}{\partial \alpha}> + <\vec{l}\cdot\vec{F}a\sin\beta> - (\frac{\partial \bar{p}}{\partial \theta})^{-1}\frac{1}{a\sin\beta}\frac{\partial}{\partial \beta}[\frac{\partial \bar{p}}{\partial \theta}<(u-w)^{\bullet}_\beta\, g^{\bullet}_{az}>\sin\beta]$$
$$-<\vec{l}\cdot[\frac{d_a\vec{W}_{oa}}{dt} - \vec{\Omega}x(\vec{\Omega}x\vec{r})]a\sin\beta> + <\frac{d\vec{k}_o}{dt}\cdot\vec{g}_a - \vec{k}_o\cdot(\vec{\Omega}x\vec{g}_a)>$$
$$-(\frac{\partial \bar{p}}{\partial \theta})^{-1}\frac{\partial}{\partial \theta}<\frac{\partial \bar{p}}{\partial \theta}g^{\bullet}_{az}\dot{\theta}^{\bullet}> \tag{4}$$

and

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{\theta}{c_p T}Q. \tag{5}$$

The respective terms on the right-hand side of (4) are the pressure torque, frictional torque, horizontal divergence of eddy angular momentum transport, inertial torque, translational source and the vertical divergence of eddy angular momentum transport (see list of symbols in Appendix D).

The definitions of storm absolute angular momentum and its time rate of change are given in Appendix A. The mass-weighted azimuthal averaged symbol (the circumflex $\wedge$or $<\ >$) and its deviation are defined in Appendix C.

The hydrostatic equation in isentropic coordinates in terms of the Montgomery streamfunction ψ is

$$\frac{\overline{\partial \psi}}{\partial \theta} = c_p(\frac{\bar{p}}{p_{oo}})^\kappa. \tag{6}$$

With a multiplication of (B.6) in Appendix B by $(a\ \ \sin\beta)^3$, the quasi-gradient wind equation in terms of the vertical component of storm absolute angular momentum is

$$\hat{g}_{az}{}^2 = -\frac{R_\beta}{a}(\frac{\partial \psi}{\partial \beta} - \frac{\partial \bar{\Psi}}{\partial \beta}) + \frac{\tilde{f}^2}{4}a^4\sin^4\beta, \tag{7}$$

where

$$\frac{\partial \bar{\Psi}}{\partial \beta} = (\int_{\theta_S}^{\theta_T}\int_0^{2\pi}\rho J_\theta \frac{\partial \psi}{\partial \beta}d\alpha d\theta) / (\int_{\theta_S}^{\theta_T}\int_0^{2\pi}\rho J_\theta d\alpha d\theta), \tag{8}$$

is a mean radial pressure gradient force which is a function of β only, and

$$R_\beta = -(a\sin\beta)^3; \tilde{f} = \Omega(\sin\varphi_o + \sin\varphi). \tag{9}$$

The azimuthally averaged hydrostatic mass distribution is expressed by

$$\overline{\rho J_\theta} = -\frac{1}{g}\frac{\overline{\partial p}}{\partial \theta}. \tag{10}$$

III. A DIAGNOSTIC EQUATION FOR THE AZIMUTHALLY-AVERAGED MASS-WEIGHTED STREAMFUNCTION OF THE RADIAL-VERTICAL CIRCULATION WITHIN A BALANCED VORTEX

The purpose of this section is to derive a diagnostic equation from (2) to (10) for the storm relative radial-vertical circulation within cyclones. This will be accomplished through the following four subsections.

1. *A Diagnostic Relation Involving the Radial and Diabatic Advections of Pressure and Torques in Isentropic Coordinates*

In this sudsection, two expressions for the tendency of the thermal wind in terms of the vertical component of absolute angular momentum will be derived in order to remove the tendency of angular momentum in (3) and to obtain a diagnostic relation involving the radial and diabatic advections of pressure and torques in isentropic coordinates. The first expression is for the tendency of the thermal wind within a balanced vortex in relation to the radial and diabatic advections of pressure in isentropic coordinates. The derivation starts with taking the vertical and time differentitation of (7) (Yuan, 1990)

$$\frac{\delta}{\delta t}\left(\frac{\partial}{\partial\theta}\hat{g}_{az}{}^2\right) = -R_\beta\frac{\delta}{\delta t}\left\{\frac{\partial}{\partial\theta}\left[\frac{1}{a}\overline{\frac{\partial\psi}{\partial\beta}} + \overline{(\rho J_\theta)'\frac{1}{a}\left(\frac{\partial\psi'}{\partial\beta}\right)}/\overline{(\rho J_\theta)}\right]\right\}. \quad (11)$$

With the use of (6), (11) becomes

$$\frac{\delta}{\delta t}\left(\frac{\partial}{\partial\theta}\hat{g}_{az}{}^2\right) = -R_\beta\frac{\delta}{\delta t}\frac{\partial}{\partial\beta}\left[\frac{c_p}{a}\overline{\left(\frac{p}{p_{oo}}\right)^\kappa}\right] - \frac{\delta}{\delta t}\frac{\partial}{\partial\theta}\left[R_\beta\overline{(\rho J_\theta)'\frac{1}{a}\left(\frac{\partial\psi'}{\partial\beta}\right)}/\overline{(\rho J_\theta)}\right]. \quad (12)$$

Following Gallimore and Johnson's analysis (Appendix E), $\overline{p^\kappa}$ is approximately expressed by $\overline{p}^\kappa$ based on the tropospheric structure of cyclones. With this approximation, (12) is rewritten as

$$\frac{\delta}{\delta t}\left(\frac{\partial}{\partial\theta}\hat{g}_{az}{}^2\right) \cong -\frac{R_\beta}{a}\frac{\partial}{\partial\beta}\left[\alpha_*\frac{\delta\overline{p}}{\delta t}\right] - \frac{\delta}{\delta t}\frac{\partial}{\partial\theta}\left[R_\beta\overline{(\rho J_\theta)'\frac{1}{a}\left(\frac{\partial\psi'}{\partial\beta}\right)}/\overline{(\rho J_\theta)}\right], \quad (13)$$

where α_* is equal to $\kappa c_p(\overline{p})^{\kappa-1}(p_{oo})^{-\kappa}$. Eq. 13 relates the tendency of the thermal wind within the azimuthally averaged mass-weighted gradient balanced vortex with the radial derivative of the tendency of the azimuthally averaged pressure and the tendency of the vertical derivative of a covariance between the azimuthally averaged deviations of the mass and radial pressure gradient force.

The next step involves the elimination of the pressure tendency in (13) with the use of hydrostatic assumption. Following Gallimore and Johnson (1981), the azimuthally averaged mass distribution for the vortex is defined by

$$\overline{m}(\beta,\theta,t) = \int_\theta^{\theta_T}\overline{\rho J_\theta}\,d\theta. \quad (14)$$

This property is a monotonic and unique function of θ with $\overline{\rho J_\theta}$ being positive. Under the hydrostatic assumption and the condition that $\overline{p}(\theta_T)$ vanishes, the azimuthally averaged pressure is defined from the azimuthally averaged mass distribution by

$$\overline{m}(\beta,\theta,t) = -\frac{1}{g}\int_\theta^{\theta_T}\frac{\partial\overline{p}}{\partial\theta}d\theta = \overline{p}(\beta,\theta,t)/g. \quad (15)$$

Since the relations between $\overline{p},\overline{m}$ and θ are unique and monotonic, the pressure tendency within the quasi-Lagrangian azimuthally averaged structure (after Gallimore and Johnson, 1981) is given by

$$\frac{\delta\overline{p}}{\delta t} = -\frac{(\hat{u}-\hat{w})_\beta}{a}\frac{\partial\overline{p}}{\partial\beta} - \dot{\theta}\frac{\partial\overline{p}}{\partial\theta} + \hat{\omega}_p. \quad (16)$$

The substitution of (16) into (13) gives

$$\frac{\delta}{\delta t}\left(\frac{\partial}{\partial \theta}\hat{g}_{az}{}^{2}\right) \cong -\frac{R_\beta}{a}\frac{\partial}{\partial \beta}\{\alpha_*[-\frac{(\hat{u}-\hat{w})_\beta}{a}\frac{\partial \overline{p}}{\partial \beta}-\hat{\dot{\theta}}\frac{\partial \overline{p}}{\partial \beta}+\hat{\omega}_p]\}$$
$$-\frac{\partial}{\partial \theta}\frac{\delta}{\delta t}[R_\beta\overline{(\rho J_\theta)'\frac{1}{a}(\frac{\partial \psi}{\partial \beta})'}/\overline{(\rho J_\theta)}]. \tag{17}$$

Eq. 17 is the first expression for the tendency of the thermal wind within a balanced vortex in relation to the radial and diabatic advections of pressure in isentropic coordinates.

The derivation of the second expression for the tendency of the thermal wind in relation to the vertical derivative of torques starts with the advective form of the azimuthally averaged mass–weighted absolute angular momentum equation obtained through the combination of (2) and (3)

$$\frac{\delta \hat{g}_{az}}{\delta t}+\frac{(\hat{u}-\hat{w})_\beta}{a}\frac{\partial \hat{g}_{az}}{\partial \beta}+\hat{\dot{\theta}}\frac{\partial \hat{g}_{az}}{\partial \theta}=\hat{F}. \tag{18}$$

The relation between the tendency of the thermal wind and the vertical derivative of torques is determined by the vertical differentiation of the product of $\hat{g}_{az}$ and (18). The result is

$$\frac{\delta}{\delta t}\left(\frac{\partial}{\partial \theta}\hat{g}_{az}{}^{2}\right)=-\frac{\partial}{\partial \theta}[2\hat{g}_{az}(\hat{u}-\hat{w})_\beta\frac{1}{a}\frac{\partial \hat{g}_{az}}{\partial \beta}]+\frac{\partial}{\partial \theta}[2\hat{g}_{az}(\hat{F}-\hat{\dot{\theta}}\frac{\partial \hat{g}_{az}}{\partial \theta})]. \tag{19}$$

A diagnostic equation is given by the combination of (17) and (19)

$$\frac{\partial}{\partial \theta}[2\hat{g}_{az}(\hat{u}-\hat{w})_\beta G_\beta]+\frac{R_\beta}{a}\frac{\partial}{\partial \beta}\{\alpha_*[-\frac{(\hat{u}-\hat{w})_\beta}{a}\frac{\partial \overline{p}}{\partial \beta}-\hat{\dot{\theta}}\frac{\partial \overline{p}}{\partial \theta}+\hat{\omega}_p]\}$$
$$=-\frac{\partial}{\partial \theta}[2\hat{g}_{az}(\hat{F}-\hat{\dot{\theta}}\frac{\partial \hat{g}_{az}}{\partial \theta})]-\frac{\partial}{\partial \theta}\frac{\delta}{\delta t}[R_\beta\overline{(\rho J_\theta)'\frac{1}{a}(\frac{\partial \psi'}{\partial \beta})'}/\overline{\rho J_\theta}], \tag{20}$$

where

$$G_\beta=-\frac{1}{a}\frac{\partial}{\partial \beta}(\hat{g}_{az}). \tag{21}$$

Eq. 20 is the diagnostic equation relating the radial and diabatic advections of the azimuthally averaged pressure with vertical derivative of torques in isentropic coordinates.

2. *A Diagnostic Equation Relating the Radial and Vertical Advections of Potential Temperature with Vertical Derivative of Torques*

In order to simplify the process of rewriting (20) into a form similar to the Eliassen's result, (20) is transformed from (β,θ,t) coordinates into the azimuthally averaged mass coordinates $(\Phi(\beta),\overline{p}(\beta,\theta,t),t)$ with $\overline{p}(\beta,\theta,t)=g\overline{m}(\beta,\theta,t)=g\int_\theta^{\theta_T}\overline{\rho J_\theta}\,d\theta$ by using the following relationship (Gallimore and Johnson, 1981)

$$\frac{\partial \overline{p}}{\partial \beta_\theta}=-\frac{\partial \overline{p}}{\partial \theta}\frac{\partial \theta}{\partial \beta_{\overline{p}}}, \tag{22}$$

$$\frac{\partial(\ )}{\partial \beta_\theta}=\frac{\partial(\ )}{\partial \beta \overline{p}}+\frac{\partial \overline{p}}{\partial \beta_\theta}\frac{\partial(\ )}{\partial \overline{p}}, \tag{23}$$

$$\frac{\partial(\ )}{\partial \theta}=\frac{\partial \overline{p}}{\partial \theta}\frac{\partial(\ )}{\partial \overline{p}}, \tag{24}$$

$$\frac{\partial(\ )}{\partial \Phi}=\frac{1}{a}\frac{\partial(\ )}{\partial \beta \overline{p}}. \tag{25}$$

With the use of the following implied thermal wind expression (Appendix B)

$$2\hat{g}_{az}\frac{\partial}{\partial\theta}(\hat{g}_{az}) = -\frac{\alpha_* R_\beta}{a}(\frac{\partial\overline{p}}{\partial\beta}) - \frac{\partial}{\partial\theta}\overline{[(\rho J_\theta)'\frac{1}{a}(\frac{\partial\psi}{\partial\beta})' / \overline{(\rho J_\theta)}]})R_\beta, \tag{26}$$

(20) becomes (Yuan, 1990)

$$\begin{aligned}&\frac{\partial}{\partial\Phi}(\alpha_*\frac{\partial\theta}{\partial\overline{p}}\hat{\omega}_p) + \frac{\partial}{\partial\Phi}[\alpha_*\frac{\partial\theta}{\partial\Phi}(\hat{u}-\hat{w})_\beta] - \frac{\partial}{\partial\overline{p}}(\alpha_*\frac{\partial\theta}{\partial\Phi}\hat{\omega}_p)\\ &-\frac{\partial}{\partial\overline{p}}[2\hat{g}_{az}G_\beta(\hat{u}-\hat{w})_\beta / (a\sin\beta)^3] - \frac{\partial}{\partial\overline{p}}[\alpha_*\frac{\partial\overline{p}}{\partial\theta}(\frac{\partial\theta}{\partial\Phi})^2(\hat{u}-\hat{w})_\beta]\\ =&\frac{\partial}{\partial\overline{p}}\{(2\hat{g}_{az}\hat{F}) / (a\sin\beta)^3 - \frac{\delta}{\delta t}\overline{[(\rho J_\theta)'\frac{1}{a}(\frac{\partial\psi}{\partial\beta})' / \overline{(\rho J_\theta)}]}\\ &-\hat{\dot{\theta}}\frac{\partial}{\partial\theta}\overline{[(\rho J_\theta)'\frac{1}{a}(\frac{\partial\psi}{\partial\beta})' / \overline{(\rho J_\theta)}]}\} + \frac{\partial}{\partial\Phi}(a_*\hat{\dot{\theta}}).\end{aligned} \tag{27}$$

Eq. 27 is the diagnostic equation relating the radial and vertical advections of potential temperature with the vertical derivative of torques in the mass coordinates $(\Phi,\overline{p},t)$.

3. *The Introduction of a Streamfunction of the Azimuthally Averaged Mass–weighted Radial–vertical Circulation within a Balanced Vortex*

Since two unknowns $[(\hat{u}-\hat{w})$ and $\hat{\omega}_p]$ for the radial–vertical circulation are involved in (27), a streamfunction of the azimuthally averaged mass–weighted radial–vertical circulation within a balance vortex will be introduced in order to reduce the unknowns to one. For this purpose the tendency of mass in the continuity equation will be removed through the differentiation of (16) with respect to θ

$$\frac{\delta}{\delta t}(\frac{\partial\overline{p}}{\partial\theta}) = -[\frac{1}{a}\frac{\partial}{\partial\theta}(\hat{u}-\hat{w})_\beta]\frac{\partial\overline{p}}{\partial\beta} - \frac{(\hat{u}-\hat{w})_\beta}{a}\frac{\partial}{\partial\beta}(\frac{\partial\overline{p}}{\partial\theta}) - \frac{\partial}{\partial\theta}(\hat{\dot{\theta}}\frac{\partial\overline{p}}{\partial\theta}) + \frac{\partial}{\partial\theta}\hat{\omega}_p. \tag{28}$$

The substitution of (28) into (2) gives

$$\frac{\partial\hat{\omega}_p}{\partial\theta} + \frac{1}{a\sin\beta}\frac{\partial}{\partial\beta}[\frac{\partial\overline{p}}{\partial\theta}(\hat{u}-\hat{w})_\beta\sin\beta] - \frac{1}{a}\frac{\partial}{\partial\theta}[(\hat{u}-\hat{w})_\beta\frac{\partial\overline{p}}{\partial\beta} - \frac{(\hat{u}-\hat{w})_\beta}{a}\frac{\partial}{\partial\beta}(\frac{\partial\overline{p}}{\partial\theta}) = 0. \tag{29}$$

According to the transformation relationships (22)–(25), (29) becomes (Yuan, 1990)

$$\frac{\partial\hat{\omega}_p}{\partial\overline{p}} - \frac{1}{\sin\beta}\frac{\partial}{\partial\Phi}[(\hat{u}-\hat{w})_\beta\sin\beta] = 0. \tag{30}$$

From (30), a streamfunction can be defined as

$$(\hat{u}-\hat{w}) = \frac{1}{\sin\beta}\frac{\partial S}{\partial\overline{p}}, \tag{31}$$

$$\hat{\omega}_p = -\frac{1}{\sin\beta}\frac{\partial S}{\partial\Phi}, \tag{32}$$

where S is the streamfunction of azimuthally averaged mass–weighted radial–vertical circulation within a balance vortex.

4. *The Diagnostic Equation for the Azimuthally Averaged Mass–weighted Streamfunction of the Radial–vertical Circulation within a Balanced Vortex*

With the streamfunction defined by (31) and (32), (27) becomes

$$\frac{\partial}{\partial \Phi}(A\frac{\partial S}{\partial \Phi} + B\frac{\partial S}{\partial \bar{p}}) + \frac{\partial}{\partial \bar{p}}(B\frac{\partial S}{\partial \Phi} + C\frac{\partial S}{\partial \bar{p}})$$

$$= \frac{\partial}{\partial \bar{p}}\{(2\hat{g}_{az}\hat{F})/(a\sin\beta)^3 - \frac{\delta}{\delta t}[\overline{(\rho J_\theta)'\frac{1}{a}(\frac{\partial \psi}{\partial \beta})'}/\overline{(\rho J_\theta)}]$$

$$-\hat{\dot{\theta}}\frac{\partial}{\partial \theta}[\overline{(\rho J_\theta)'\frac{1}{a}(\frac{\partial \psi}{\partial \beta})'}/\overline{(\rho J_\theta)}]\} + \frac{\partial}{\partial \Phi}(\alpha_* \hat{\dot{\theta}}), \tag{33}$$

where

$$A = -\frac{\alpha_*}{\sin\beta}\frac{\partial \theta}{\partial \bar{p}}, \tag{34}$$

$$B = \frac{\alpha_*}{\sin\beta}\frac{\partial \theta}{\partial \Phi}, \tag{35}$$

$$C = -[\frac{2g_{az}G_\beta}{(a^3\sin^4\beta)} + \frac{\alpha_*}{\sin\beta}\frac{\partial \bar{p}}{\partial \theta}(\frac{\partial \theta}{\partial \Phi})^2]. \tag{36}$$

Coefficient A expresses the static stability, B the degree of baroclinity, and C the inertial stability. Eq. 33 is the diagnostic equation for the forcing of the azimuthally averaged mass-weighted radial-vertical circulation within cyclones. The property $\hat{F}$ in the first right hand side (r.h.s.) term in (33) represents the forcing of the circulation by pressure torque, frictional torque, inertial torque, divergence of eddy angular momentum transport and other effects (see equation 4).

The classification criterion of second order differential equation like (33) is expressed by

$$\delta^2 = AC - B^2. \tag{37}$$

The substitution (34)–(36) into (37) yields

$$\delta^2 = \frac{\alpha_*}{a^3\sin^5\beta}(-\frac{\partial \theta}{\partial \bar{p}})\frac{1}{a}\frac{\partial}{\partial \beta}(\hat{g}_{az}{}^2). \tag{38}$$

With

$$-\frac{\partial \theta}{\partial \bar{p}} > 0 \quad \text{and} \quad \frac{\partial}{\partial \beta}(\hat{g}_{az}{}^2) > 0, \tag{39}$$

δ^2 is positive definite and (33) is an elliptic equation. It is important to realize that the conditions in (39) require only that the azimuthally averaged pressure decreases with potential temperature and that the azimuthally averaged mass-weighted vertical component of absolute angular momentum square increases with radius along an isentropic surface. Therefore, local static and inertial instabilities could exist while average conditions as expressed by (39) may still be satisfied.

VI. A BRIEF INTERPRETATION OF THE PHYSICAL PROCESSES INVOLVED IN FORCING THE RADIAL-VERTICAL CIRCULATION WITHIN EXTRATROPICAL CYCLONES

Since (33) is formally equivalent to Eliassen's result, the theoretical conclusions from his study for a closed system will be applied to the following discussion that 1) for a vortex, motion away from (toward) the axis of rotation and along isentropic surfaces is forced by positive (negative) torques; 2) diabatic heating (cooling) forces motion upward (downward) through isentropic surfaces along surfaces of constant absolute angular momentum and 3) for a given value of heating and / or sources and sinks of angular momentum by torques, the weaker the hydrodynamic stability of the vortex is, the stronger the radial–vertical circulation. The effects of torques and diabatic heating on the storm relative mass and angular momentum transport have been discussed in previous budget studies (Johnson, 1974; Johnson and Downey, 1975b; Katzfey, 1978; Schneider, 1986; Johnson and Hill, 1987). A brief review is presented in this section.

1. *Pressure Torque*

Johnson and Downey (1975b) discussed the physical basis for the azimuthal pressure torque in various coordinate systems. Their results indicate that the azimuthally averaged azimuthal pressure torque vanishes in the Cartesian and isobaric coordinates but not in isentropic coordinates. Note that the azimuthally averaged mass–weighted pressure torque in isentropic coordinates (Johnson and Downey, 1975b) is expressed by

$$
\begin{aligned}
- < \frac{\partial \psi}{\partial \alpha} > &= - \overline{\rho J_\theta \frac{\partial \psi}{\partial \alpha}} \quad / \overline{\rho J_\theta} \\
&= \{ \frac{c_p \theta}{g(1+\kappa) p_{oo}{}^{\kappa}} \overline{\frac{\partial}{\partial \alpha} (\frac{\partial}{\partial \theta} p^{1+\kappa})} + \frac{\partial}{\partial \theta} \overline{p \frac{\partial h}{\partial \alpha}} \} / \overline{\rho J_\theta},
\end{aligned} \tag{40}
$$

where h represents the height of the isentropic surface, $\partial h / \partial \alpha$ is the slope of an isentropic surface in the azimuthal direction and $p(\partial h / \partial \alpha)$ is the azimuthal component of pressure stress vector $[\vec{P} = p\vec{n} = p(\partial h / \partial \alpha)\vec{i} - p(\partial h / \partial \theta)\vec{k}]$ of an upper domain on a lower domain through the inclined isentropic surface which separates these two domains (Katzfey, 1978, 1983; Czarnetzki and Johnson, 1995). In lower atmosphere where isentropic surfaces intersect with the earth's surface, Lorenz's convention (Lorenz, 1955) is applied to the calculation of pressure torque. Following Lorenz's convention, the pressure and geopotential height on the "underground" isentropic surfaces ($\theta \leqslant \theta_S$, θ_S is the surface potential temperature) are set to their values at earth's surface. Therefore, the azimuthal integration of first term in (40) is zero due to the cyclic continuity in the azimuthal direction. The azimuthally averaged mass–weighted pressure torque in isentropic coordinates becomes

$$
- < \frac{\partial \psi}{\partial \alpha} > = \frac{\partial}{\partial \theta} \overline{(p \frac{\partial h}{\partial \alpha})} / \overline{(\rho J_\theta)}. \tag{41}
$$

Expression (41) states that mass–weighted azimuthally averaged pressure torque results from the difference between the azimuthally averaged azimuthal component of pressure stress on upper and lower inclined isentropic surfaces of a volume element. Due to this torque, angular momentum is transferred across inclined isentropic surfaces by pressure stresses, just as angular momentum is transferred across mountain surfaces by viscous stresses.

This expression given by (41) also illustrates the classical result that there are no net internal sources of absolute angular momentum by pressure force since the vertical integration of (41) reduces to pressure stresses acting on the boundaries. Therefore, over a uniform earth's surface net positive and negative torques (or source and sink of absolute angular momentum) within isentropic layers must offset each other, which implies that there exists a non–convective momentum transfer through inclined isentropic surfaces duo to pressure torque (Johnson and Downey, 1975b).

The non–convective transfer of absolute angular momentum across the inclined isentropic surfaces and the forcing of the radial motion by pressure torque occur simultaneously within cyclones with a non–axisymmetric baroclinic structure. An example of vertical distribution of areally integrated pressure torque over areas with radii of 6° (solid) and 9° (dashed) of equivalent latitude respectively at 1200 UTC 23 within a baroclinic vortex that occurred in April 1968 over the midwestern United States is shown in Fig. 2 with negative pressure torque in the lower valued isentropic layers and positive pressure torque in the higher valued isentropic layers. According to the one to one correspondence between the pressure torque and the geostropic radial mass transport

$$\text{pressure torque} = -<\frac{\partial\psi}{\partial\alpha}> = \overline{\rho J_\theta fa\sin\beta(U_g)_\beta} \,/\, \overline{\rho J_\theta}, \tag{42}$$

their results of mean geostrophic lateral mass transport confirm Eliassen's (1951) conclusion that inward mass transport $[\rho J_\theta(U_g)_\beta < 0]$ is forced by negative pressure torque in lower valued isentropic layers and outward $[\rho J_\theta(U_g)_\beta > 0]$ positive pressure torque in high valued isentropic layers.

2. *Frictional Torque*

Similar to the pressure torque, the frictional torque $\vec{l}\cdot\vec{F}a\sin\beta$ involves angular momentum exchange in that viscous stresses transfer angular momentum across isentropic / earth surfaces and across lateral boundaries of a storm volume. Since the viscous stresses acting on lateral boundaries are presumed small, the net frictional torque primarily involves the sink of storm angular momentum to the earth by viscous stressed.

In the absolute angular momentum budget study of a Midwest cyclone of April 1968 (Fig. 3), the turbulent characteristics of the boundary layer are related to the large-scale synoptic parameters, such as the geostrophic wind and drag coefficient (Lettau, 1959). As it is shown in Fig. 3, the frictional torque is negative throughout the life cycle of the storm, which will force inflow in lower layers.

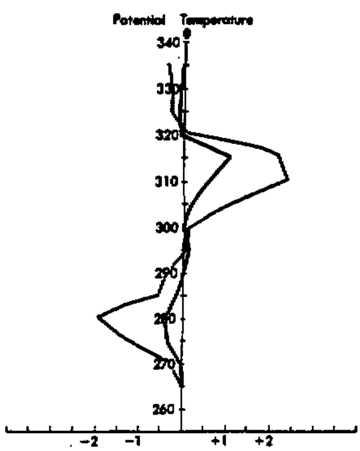

Fig. 2. Vertical profiles of pressure torque (10^{18}kg m^{2}s^{-2}) evaluated for 5 K isentropic layers of storm volumes with radii equivalent to 6° (solid line) and 9° (dashed line) of latitude (Johnson and Downey, 1975b).

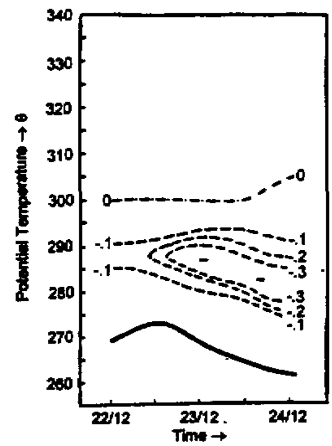

Fig. 3. Vertical-time section of frictional torque (10^{18}m^{2}s^{-2}) for the time period of 1200 UTC 24 April 1968 (Johnson and Downey, 1976).

3. *The Divergence of Eddy Angular Momentum Transport (Eddy Mode)*

The eddy angular momentum transport is associated with the covariance between the mass–weighted deviations of radial component of wind and vertical component of absolute angular momentum. The forcing of the radial–vertical circulation due to the asymmetries of the wind field is associated with the radial divergence of radial eddy angular momentum transport

$$-\left(\frac{\partial \bar{p}}{\partial \theta}\right)^{-1}\frac{1}{a\sin\beta}\frac{\partial}{\partial \beta}\frac{\partial \bar{p}}{\partial \theta}<(u-w)^{*}_{\beta}\, g^{*}_{az}>\sin\beta]$$
$$=-\left(\frac{\partial \bar{p}}{\partial \theta}\right)^{-1}\frac{1}{a\sin\beta}\frac{\partial}{\partial \beta}\left[\overline{\frac{\partial p}{\partial \theta}(u-w)^{*}_{\beta}\, g^{*}_{az}\sin\beta}\right]. \tag{43}$$

During the development of an extratropical cyclone, an S–shaped pattern in the wind field in the upper troposphere is observed (Wash, 1978; Hale, 1983; Schneider, 1986) due to the adjustment of upper atmospheric wind field to the low–level baroclinic zone through the thermal wind. Figure 4 shows the streamline and isotach analyses on 315 K (near 300 hPa) for an extratropical cyclone that occurred over the United States in April 1979 (Schneider, 1986). This synoptic situation is simplified by Johnson et al. (1981) in Fig. 5 which shows that in the exit region of the southern jet an inflow deviation is associated with positive deviation of angular momentum. In the entrance region of the northern jet, an outflow deviation is associated with negative deviation of angular momentum. Therefore, the eddy angular momentum transport is convergent due to the S–shaped upper–level wind pattern. According to Eliassen (1951), this positive eddy mode results in outflow in this layer.

The importance of the convergence of eddy angular momentum transport in the development of cyclones has been emphasized by Wash (1978), and Johnson and Hill (1987).

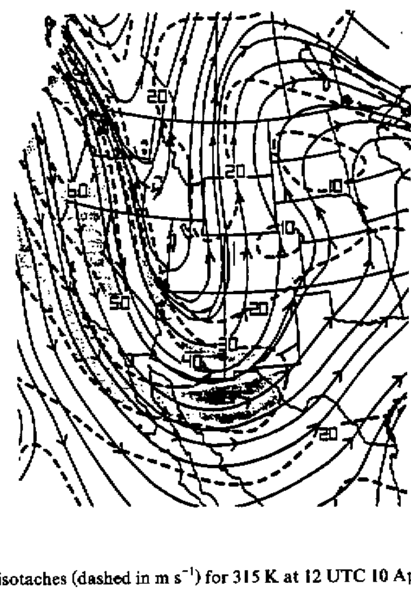

Fig. 4. Streamlines and isotaches (dashed in m s^{-1}) for 315 K at 12 UTC 10 April 1979 (Schneider, 1986).

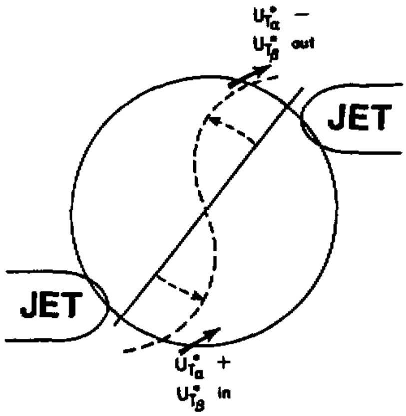

Fig. 5. Schematic of upper level eddy lateral angular momentum transport that is associated with the contortion and rotation of the thermal field (Johnson et al., 1981).

Noticing that the azimuthally averaged absolute angular momentum within a cyclone increases with radius, they point out that the increase of the angular momentum due to this eddy mode in the isentropic layers in which the mean mode of angular momentum transport is outward *is the excess of the convergence of eddy mode over the divergence of mean mode.*

4. *Inertial Torque*

Within the balance of storm angular momentum, the inertial torque stems from the acceleration / deceleration of the storm axis with respect to an absolute framework. Through an asymmetric mass distribution, the inertial torque is expressed by

$$
\begin{aligned}
& - < \vec{l} \cdot [\frac{d_a \overrightarrow{W}_{oa}}{dt} - \vec{\Omega} x(\vec{\Omega} x \vec{r})] > a\sin\beta \\
= & - < \vec{l} \cdot \frac{d\overrightarrow{W}_o}{dt} > a\sin\beta - < \vec{l} \cdot (2\vec{\Omega} x \overrightarrow{W}_o) > a\sin\beta \\
& + < \vec{l} \cdot \vec{\Omega} x[\vec{\Omega} x(\vec{r} - \vec{r}_o)]\} > a\sin\beta. \qquad (44)
\end{aligned}
$$

The first r.h.s. term in (44) is associated with the earth–relative acceleration / deceleration of the cyclone's movement. In an example shown in Fig. 6a, the acceleration vector $(d\overrightarrow{W}_o / dt)$ points to the west–southwest. The azimuthal component of storm acceleration vector is positive through most of the cold sector (lower valued isentropic layer) and negative through most of the warm sector (higher valued isentropic layer). Within a typical two–layer baroclinic structure which is bounded by θ_B and θ_T at the bottom and top of the storm volume respectively and divided by an isentropic surface θ_M (Fig. 6b), the azimuthally averaged mass–weighted azimuthal component of storm acceleration vector is positive (negative) due to more mass or thicker isentropic layer in lower (upper) domain.

The second term in (44) deals with azimuthally averaged mass–weighted azimuthal component of Coriolis force associated with earth–relative velocity of the storm center. In the example shown in Fig. 7, a Midwest cyclone moved northeastward with almost constant

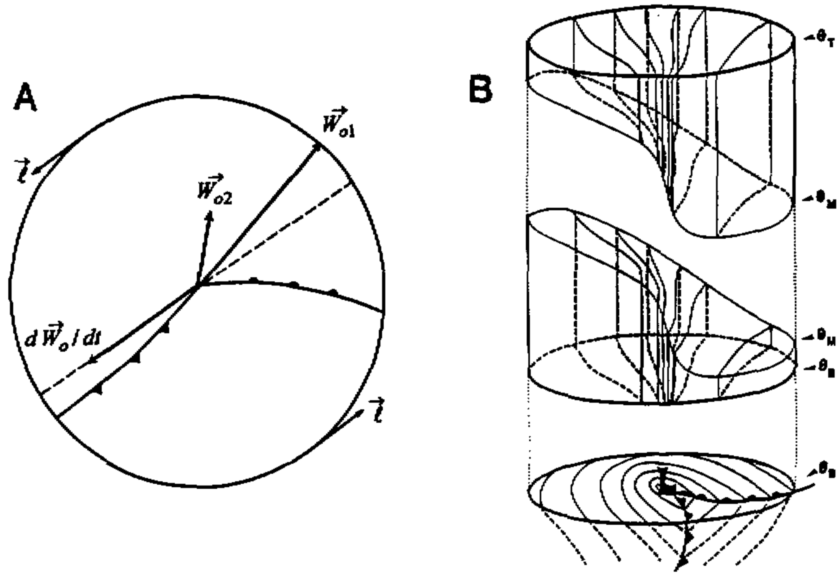

Fig. 6. Schematic of (a) acceleration of a cyclone. $\vec{W}_{o1}$, $\vec{W}_{o2}$ is the earth-relative velocity of a cyclone system at t_1 (t_2), $d\vec{W}_o/dt$ is the acceleration vector and $\vec{l}$ is the unit vector of azimuthal coordiante and (b) a simplified two-layer mass distribution bounded by θ_B and θ_T and divided by θ_M.

earth-relative velocity during the time period from 1200 UTC 23 to 1200 UTC 24 April 1968. Within this period of time, the second term will be the dominant term. Due to the northeastward movement of the cyclone in the Northern Hemisphere, the dominant direction of vector $-2\vec{\Omega}x\vec{W}_o$ is southeastward and $-\vec{l}\cdot(2\vec{\Omega}x\vec{W}_o)$ is negative in the leading half of the storm volume and positive in the trailing half of the storm volume. Since the mean orientation of the cold front shown in Fig. 7 is southeastward from the low center during this time period, more hydrostatic mass due to the cold dome behind the cold front is located in the trailing half of the volume in the lower valued isentropic layer. Thus the azimuthally averaged mass-weighted second term in (44) is positive in the lower valued isentropic layer.

The third term is associated with the contribution of the earth's rotation to the angular momentum about the local vertical axis of the cyclone. This term is cancelled by the last term of (47) which is the expression of the forcing term associated with the tilting of storm angular momentum towards or away from the local vertical axis (see next section).

Previous angular momentum budget studies (Johnson and Downey, 1975b; Katzfey, 1978; Schneider, 1986; Johnson and Hill, 1987) show that like the profile of pressure torque, there exist positive and negative counterparts in the vertical profiles of inertial torque. This condition, that the inertial torques force inward and outward lateral branches of mass circulation in different isentropic layers, implies that the processes may also be viewed as a mode by which absolute angluar momentum is transferred through the isentropic layers of a vortex (Johnson and Downey, 1975b).

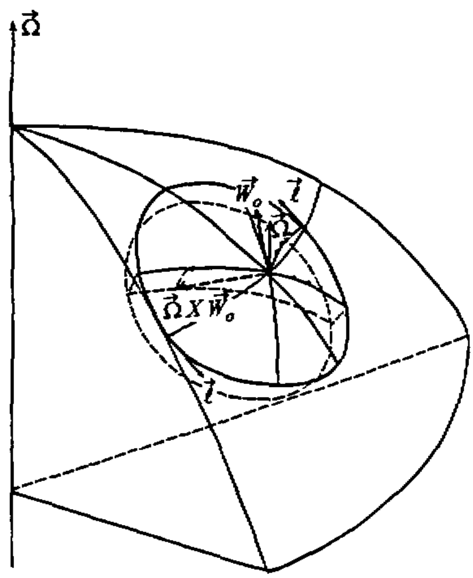

Fig. 7. Schematic of inertial torque due to $2\vec{\Omega} X \vec{W}_o$, where $\vec{W}_o$ is the velocity of the storm center.

5. *The Tilting of Angular Momentum towards or away from the Local Vertical Axis due to Storm Movement*

The tilting of storm angular momentum towards or away from the local vertical axis is represented by

$$< \frac{d\vec{k}_o}{dt} \cdot \vec{g}_a > - < \vec{k}_o \cdot (\vec{\Omega} x \vec{g}_a) > = < \frac{\vec{W}_o}{r_o} \cdot \vec{g}_a > - < \vec{k}_o \cdot (\vec{\Omega} x \vec{g}_a) > . \tag{45}$$

The first r.h.s. term in (45) deals with the earth-relative movement of the coordinates along the spherical earth's surface and the second r.h.s. term in (45) deals with the rotation of the earth. Only the horizontal component of absolute angular momentum contributes to the tilting term. Since absolute angular momentum is expressed by (Johnson and Downey, 1975b)

$$\begin{aligned} \vec{g}_a &= (\vec{r} - \vec{r}_o) x (\vec{U}_a - \vec{W}_{oa}) = (\vec{r} - \vec{r}_o) x [(\vec{U} - \vec{W}_o) + \vec{\Omega} x (\vec{r} - \vec{r}_o)] \\ &= (\vec{r} - \vec{r}_o) x (\vec{U} - \vec{W}_o) + [(\vec{r} - \vec{r}_o) \cdot (\vec{r} - \vec{r}_o)] \vec{\Omega} - [(\vec{r} - \vec{r}_o) \cdot \vec{\Omega}] (\vec{r} - \vec{r}_o), \end{aligned} \tag{46}$$

the expansion of (45) becomes (Schneider, 1986)

$$\begin{aligned} < \frac{\vec{W}_o}{r} \cdot \vec{g}_a - \vec{k}_o \cdot (\vec{\Omega} x \vec{g}_a) > &= < \vec{W}_o \cdot \{ [\vec{\Omega} (\vec{r} - \vec{r}_o)^2] - (\vec{r} - \vec{r}_o) [(\vec{r} - \vec{r}_o) \cdot \vec{\Omega}] \\ &+ [(\vec{r} - \vec{r}_o) x \vec{U}] \} / \vec{r}_o > - < \vec{k}_o \cdot (\vec{r} - \vec{r}_o) > \end{aligned}$$

$$< \vec{\Omega} \cdot (\vec{U} - \vec{W}_o) > + < (\vec{k}_o \cdot \vec{U})[(\vec{r} - \vec{r}_o) \cdot \vec{\Omega}] > - < \vec{l} \cdot \{\vec{\Omega} x [\vec{\Omega} x (\vec{r} - \vec{r}_o)]\} > a\sin\beta. \quad (47)$$

In (47), the last term has the same absolute value as the third term in (44) but opposite sign, so they cancel each other. The results of several angular momentum budget studies (Wash, 1978; Hale, 1983; Schneider, 1986) show that this forcing term, which is associated with the tilting of storm angular momentum towards or away from the local vertical axis, is smaller than pressure torque and other forcing terms.

6. *Diabatic Processes*

The last r.h.s. term in (33) specifies the forcing associated with diabatic processes. Within a cyclone, upward branches are forced primarily through latent heat release within moist convection and in some cases through sensible heating in the boundary layer. Downward branches of the radial–vertical circulation are forced through the cooling due to radiative processes and evaporation.

Another forcing term associated with the covariance between the azimuthally averaged mass–weighted deviations of diabatic heating and vertical component of absolute angular momentum is the vertical eddy mode $[-(\frac{\partial \bar{p}}{\partial \theta})^{-1} \frac{\partial}{\partial \theta} (\frac{\partial \bar{p}}{\partial \theta} < \dot{\theta}^* g^*_{az} >)]$. A diagnostic study (Hale, 1983) based on areally averaged quantities indicates that the vertical eddy angular momentum transport has some contribution to the development of an intense Ohio Valley extratropical cyclone.

7. *Estimation of the Magnitude of the Forcing Terms in the Streamfunction Equation*

In this section, the orders of magnitude of four forcing terms are estimated and compared. The first term deals with pressure torque

$$\frac{\partial}{\partial \bar{p}}[-2\hat{g}_{az} < \frac{\partial \psi}{\partial \alpha} > / (a\sin\beta)^3] = \frac{\partial \theta}{\partial \bar{p}} \frac{\partial}{\partial \theta}[-2\hat{g}_{az} < \frac{\partial \psi}{\partial \alpha} > / (a\sin\beta)^3], \quad (48)$$

which is one of the dominant forcing components in the early stage of a Mediterranean cyclone of March 1982 (Johnson and Hill, 1987). The second term

$$\frac{\partial}{\partial \bar{p}}\{-\frac{\delta}{\delta t}\overline{[(\rho J_\theta)' \frac{1}{a}(\frac{\partial \psi}{\partial \beta})' / \overline{(\rho J_\theta)}]}\} = \frac{\partial \theta}{\partial \bar{p}} \frac{\partial}{\partial \theta}\{-\frac{\delta}{\delta t}\overline{[(\rho J_\theta)' \frac{1}{a}(\frac{\partial \psi}{\partial \beta})' / \overline{(\rho J_\theta)}]}\} \quad (49)$$

deals with the vertical and temporal variation of the covariance of the azimuthally averaged deviations of mass and radial pressure torque. The third term

$$\frac{\partial}{\partial \bar{p}}\{-\hat{\dot{\theta}}\frac{\partial}{\partial \theta}\overline{[(\rho J_\theta)' \frac{1}{a}(\frac{\partial \psi}{\partial \beta})' / \overline{(\rho J_\theta)}]}\} = \frac{\partial \theta}{\partial \bar{p}} \frac{\partial}{\partial \theta}\{-\hat{\dot{\theta}}\frac{\partial}{\partial \theta}\overline{[(\rho J_\theta)' \frac{1}{a}(\frac{\partial \psi}{\partial \beta})' / \overline{(\rho J_\theta)}]}\} \quad (50)$$

deals with the second order vertical derivative of the covariance of the azimuthally averaged deviations of mass and radial pressure torque. The fourth term

$$\frac{\partial}{\partial \Phi}(\alpha_* \hat{\dot{\theta}}) = \frac{1}{a}[\frac{\partial}{\partial \beta_\theta}(\alpha_* \hat{\dot{\theta}}) - \frac{\partial \bar{p}}{\partial \beta_\theta} \frac{\partial \theta}{\partial \bar{p}} \frac{\partial}{\partial \theta}(\alpha_* \hat{\dot{\theta}})] \quad (51)$$

deals with the azimuthally averaged mass–weighted diabatic heating rate.

The radial distribution of the order of the magnitude of these four terms is given in Table 1 for an oceanic cyclone that occurred over the Atlantic Ocean in February 1979. The data used in the calculations are ECMWF FGGE level III data on 1.875° latitude / longitude grid at 12-hour time interval. These data are interpolated to a storm spherical isentropic coordinate system (Johnson and Downey, 1975a) which is composed of 17 isentropic levels at 10 K intervals from 220 K to 380 K and seven concentric circles at radii of 1.5°, 3.0°, 4.5°, 6.0°, 7.5°, 9.0° and 10.5° latitude. Each circle contains 36 grid points that surround the storm center. When viewing these results, it is important to note that there exist three limitations in the calculation due to the data and the computational method. First, the spatial and temporal resolutions of the data are quite coarse compared to the spatial and temporal scales of cyclones. Second, ECMWF FGGE data result from assimilation which is the combination of observed and numerically simulated data, and third, the calculation of high order derivative quantities as (49) and (50) based on difference method may contain noise. In spite of these limitations, the order of magnitude should provide insight into the relative importance of these terms for the oceanic extratropical cyclone.

Table 1. Estimation of the Magnitude of the Forcing Terms for an Oceanic Extratropical Cyclone of February 1979

forcing terms (m $hPa^{-1}s^{-3}$)	radius 4.5°	radius 6.0°
$\frac{\partial}{\partial \bar{p}}[(-2\hat{g}_{az}<\frac{\partial \psi}{\partial \alpha}>)/(a\sin\beta)^3]$	10^{-11}	10^{-12}
$\frac{\partial}{\partial \bar{p}}\{-\frac{\delta}{\delta t_\theta}\overline{[(\rho J_\theta)'\frac{1}{a}(\frac{\partial \psi}{\partial \beta})'}/\overline{(\rho J_\theta)}]\}$	10^{-14}	10^{-14}
$\frac{\partial}{\partial \bar{p}}\{-\hat{\dot{\theta}}\frac{\partial}{\partial \theta}\overline{[(\rho J_\theta)'\frac{1}{a}(\frac{\partial \psi}{\partial \beta})'}/\overline{(\rho J_\theta)}]\}$	10^{-14}	10^{-14}
$\frac{\partial}{\partial \Phi}(\alpha\,.\,\hat{\dot{\theta}})$	10^{-12}	10^{-12}

The results in Table 1 show that for this oceanic cyclone the forcing associated with pressure torque (the first term) is three orders of magnitude larger than the forcing associated with the covariance between the azimuthally averaged deviations of the mass and radial pressure gradient force (the second and third terms) at smaller radius and two orders larger at larger radius. Table 1 also shows that the forcing associted with the azimuthally averaged mass-weighted diabatic heating (the fourth term) is one order of magnitude smaller than the first term at smaller radius and has the same order as the first term at larger radius. Since those two forcing terms associated with the covariance between the azimuthally averaged deviations of the mass and radial pressure gradient force [(49) and (50)] involve third derivatives, their diagnoses by finite differencing are not feasible due to low spatial and temporal resolutions of the original data as well as the low resolutions used in the numerical simulation.

V. SUMMARY

The results of mass and angular momentum budget studies of cyclones indicate that 1) the development of cyclones requires net import of absolute angular momentum from their environment through radial-vertical mass circulation (Johnson and Downey, 1975c; Johnson and Hill, 1987) and 2) the investigation of the forcing of the radial-vertical mass circulation within cyclones will provide insight into the physical processes responsible for the evolution of cyclones.

Through the application of a theory developed by Eliassen (1951) for the forced meridional circulation within a stable and symmetric circumpolar vortex and the theory developed by Gallimore and Johnson (1981) for the forced zonally averaged meridional circulation within a stable and asymmetric circumpolar vortex, a diagnostic equation is derived for the forced azimuthally averaged mass-weighted radial-vertical circulation within stable, asymmetric and translating extratropical and tropical cyclones. This diagnostic equation includes several isolated physical processes, which are respectively associated with pressure torque (representing the baroclinic structure of atmosphere), frictional torque, diabatic heating and eddy angular momentum transport. In addition, the impact of inertial stability, baroclinic stability and static stability, the effects of rotation of the earth, the acceleration of cyclone's movement and the tilting of local vertical axis with respect to the earth's rotation axis due to cyclone's movement over the spherical earth are also included.

The diagnostic equation derived in this paper relates the radial-vertical circulation with the sinks / sources of angular momentum (negative / positive torques) and diabatic heating. With the use of Eliassen's (1951) theory, and the results of angular momentum budget studies of cyclones (Johnson, 1974; Johnson and Downey, 1975a, b, 1976; Johnson and Hill, 1987), a brief interpretation is given for the roles of torques and heating in forcing the radial-vertical circulation in several extratropical cyclones.

With a two-layer structure of the azimuthally averaged mass transport, negative pressure torques will force inward mass transport in lower valued isentropic layers and positive pressure torques will force outward mass transport in higher valued isentropic layers. This distribution occurs in extratropical cyclones undergoing moist non-axisymmetric baroclinic development (Johnson and Downey, 1975b and Johnson and Hill, 1987). The convergence of horizontal eddy angular momentum transport in higher valued isentropic layers results from the S-shaped upper-level wind field due to the adjustment of the upper-layer winds to the rotation of the lower-layer baroclinic zone through thermal wind. The distribution of frictional torque with negative values in lower layers (Johnson and Downey, 1975c) will result in inflow in these layers according to Eliassen's (1951) theory. Other effective torques strongly depend on the movement of cyclones and asymmetries of its mass distribution with respect to the axis of rotation.

The influence of diabatic processes on the radial-vertical circulation occurs through radiation, evaporation, latent heat release within moist convection and sensible heat flux in the boundary layer. Heating is responsible for upward branches of the radial-vertical circulation and cooling for downward branches of the radial-vertical circulation.

The radial-vertical circulation in a cyclone undergoing moist development will be generated with inflow (forced by negative torques) in lower valued isentropic layer, outflow (forced by positive torques) in higher valued isentropic layer, upward motion associated with diabatic heating and downward motion associated with diabatic cooling. This radial-vertical circulation will provide the cyclone with angular momentum needed for its development (Johnson and Downey, 1976; Johnson and Hill, 1987). The occlusions of cyclones require vertical transfer of storm angular momentum. The transfer of storm absolute angular momentum across inclined isentropic surfaces is associated with four processes: nonconvective flux by pressure and viscous stresses, a virtual transfer by inertial torques and diabatic transport. With the use of Lagrangian perspective within isentropic framework, these processes are isolated and treated explicitly.

In the future, emphasis will be placed upon numerically solving the diagnostic equation

(33) through the use of the successive–over–relaxation (SOR) method. Various case studies will be carried out to test the ability of the diagnostic equation to simulate azimuthally–averaged mass–weighted radial motion within cyclones. With the use of the diagnostic equation, the consistency between transport processes and physical processes will be examined and the relatively important contribution of various physical processes to the forcing of the azimuthally–averaged mass–weighted radial–vertical circulation within different translating cyclones and in their different stages of development will be investigated. These results will be presented in companion articles.

The authors would like to thank Mr. Todd Schaack and Professor Kim Van Scoy for their scientific and editorial contributions. This research was sponsored by NASA Grant NAG5–81.

APPENDIX A

The Definition of Storm Angular Momentum and Its Time Rate of Change

Storm angular momentum defined as the moment of momentum about a moving storm axis (Johnson and Downney, 1975b) is expressed by

$$\vec{g}_a = (\vec{r} - \vec{r}_o) x (\vec{U}_a - \vec{W}_{oa}), \tag{A.1}$$

where $\vec{r}$ is a position vector with its origin at the earth's center, $\vec{r}_o$ is the position vector of the storm's center, $\vec{U}_a$ and $\vec{W}_{oa}$ are the absolute velocities of $\vec{r}$ and $\vec{r}_o$ respectively (see Fig. 1). The corresponding orthogonal right–handed set of unit vectors $\vec{m}$, $\vec{l}$, $\vec{k}$ are defined (Johnson and Downey, 1975a) as

$$\vec{k} = \vec{r} / |\vec{r}| \tag{A.2}$$

$$\vec{l} = \vec{k} x (\vec{r} - \vec{r}_o) / |\vec{k} x (\vec{r} - \vec{r}_o)| \tag{A.3}$$

$$\vec{m} = \vec{l} x \vec{k}. \tag{A.4}$$

The corresponding coordinate surfaces are the vertical plants, α = constant; the cones, β = constant; the quasi–horizontal surface, θ = constant.

Since cyclones are primarily characterized by rotation in a horizontal plane, the vertical component of vector angular momentum is selected for study. With the unite vector $\vec{k}_o$ defined (Johnson and Downey, 1975b) by the gravity force as

$$\vec{k}_o = (\nabla \varphi)_o / |(\nabla \varphi)_o|$$

the vertical component of angular momentum considered here is

$$g_{az} = \vec{k}_o \cdot \vec{g}_a = a \sin\beta (u_a - w_{oa})_\alpha, \tag{A.5}$$

while its time rate of change becomes

$$\frac{d_a g_{az}}{dt} = \frac{d_a(\vec{k}_o \cdot \vec{g}_a)}{dt} = \vec{k}_o \cdot \frac{d_a \vec{g}_a}{dt} + \frac{d_a \vec{k}_o}{dt} \cdot \vec{g}_a$$

$$= -\frac{\partial \psi}{\partial \alpha} + \vec{l} \cdot [\vec{F} - \frac{d_a \vec{W}_{oa}}{dt} + \vec{\Omega} x (\vec{\Omega} x \vec{r})] a \sin\beta + \frac{d_a \vec{k}_o}{dt} \cdot \vec{g}_a = \frac{d g_{az}}{dt}, \tag{A.6}$$

where $\vec{F}$ is the frictional force, $\psi = c_p T + gz$.

APPENDIX B

An Equation for Quasi–Gradient Wind in Terms of the Storm Absolute Angular Momentum and the Thermal Wind Expression

The assumed quasi–gradient wind equation for the storm relative circulation is expressed by

$$\frac{(u-w_o)_\alpha^2}{a\sin\beta}+\bar{f}(u-w_o)_\alpha=\frac{1}{a}(\frac{\partial\psi}{\partial\beta}-\frac{\partial\bar{\psi}}{\partial\beta}), \tag{B.1}$$

where

$$\frac{\partial\bar{\psi}}{\partial\beta}=(\int_{\theta_S}^{\theta_T}\int_0^{2\pi}\rho J_\theta\frac{\partial\psi}{\partial\beta}d\alpha d\theta)/(\int_{\theta_S}^{\theta_T}\int_0^{2\pi}\rho J_\theta d\alpha d\theta) \tag{B.2}$$

is a mean radial pressure gradient force. Adding $\bar{f}^2 a\sin\beta/4$ to both sides of (B.1) and rearranging it yields

$$\frac{1}{(a\sin\beta)^3}\{(u-w_o)_\alpha a\sin\beta+\frac{\bar{f}}{2}(a\sin\beta)^2\}^2=\frac{1}{a}(\frac{\partial\psi}{\partial\beta}-\frac{\partial\bar{\psi}}{\partial\beta})+\frac{\bar{f}^2}{4}a\sin\beta. \tag{B.3}$$

Now (B.3) will be rewritten by applying the definition of the vertical component of absolute angular momentum given by Johnson and Downey (1975b) and Wash (1978)

$$g_{az}=(u-w_o)_\alpha a\sin\beta+a^2(1-\cos\beta)(f_o+f)/2, \tag{B.4}$$

where f_0 equals $2\Omega\sin\varphi_0$ and f equals $2\Omega\sin\varphi$. With a small angle approximation, (B.3) becomes

$$\begin{aligned}g_{az}&\cong(u-w_o)_\alpha a\sin\beta+(a\sin\beta)^2(f_o+f)/4\\&=(u-w_o)_\alpha a\sin\beta+(a\sin\beta)^2\bar{f}/2,\end{aligned} \tag{B.5}$$

where $\bar{f}=(f_o+f)/2$.
With the substitution of (B.5) into (B.3), the equation for the gradient wind in terms of the storm absolute angular momentum becomes

$$\frac{g_{az}^2}{(a\sin\beta)^3}=\frac{1}{a}(\frac{\partial\psi}{\partial\beta}-\frac{\partial\bar{\psi}}{\partial\beta})+\frac{\bar{f}^2}{4}a\sin\beta. \tag{B.6}$$

The vertical derivative of (B.6) with the use of the same strategies as those in subsection under section 3 gives the following implied thermal wind expression

$$2\hat{g}_{az}\frac{\partial}{\partial\theta}(\hat{g}_{az})=-\frac{\alpha_* R_\beta}{a}(\frac{\partial\bar{p}}{\partial\beta})_\theta-\frac{\partial}{\partial\theta}[\overline{(\rho J_\theta)'\frac{1}{a}(\frac{\partial\psi}{\partial\beta})'}/\overline{(\rho J_\theta)}]R_\beta. \tag{B.7}$$

APPENDIX C

The Definition of Mass–weighted Azimuthal Average and Its Deviation

The mass–weighted azimuthal average ($\wedge$ or $<>$) and its deviation ($*$) (Johnson and Downey, 1975a; Dutton, 1976) are defined by

$$<(\ )>=(\wedge)=\overline{\rho J_\theta(\ )}/\overline{\rho J_\theta}, \tag{C.1}$$

$$(\ )^{*} = (\ ) - (\wedge) = (\ ) - <(\ )>, \tag{C.2}$$

where azimuthal average $\overline{(\ )}$ and its deviation $(\ )'$ are defined by

$$\overline{(\ )} = \frac{1}{2\pi}\int_0^{2\pi}(\ )d\alpha, \qquad (\ )' = (\ ) - \overline{(\ )}. \tag{C.3}$$

APPENDIX D

List of Symbols

Symbol	
a	radius of earth
c_p	specific heat at constant pressure
f	Coriolis parameter at latitude φ
f_0	Coriolis parameter at latitude φ_0
$\tilde{f}$	$0.5(f+f_0)$(see Appendix B)
g	acceleration due to gravity
g_{az}	vertical component of absolute angular momentum
h	height of isentropic surface
k_n	azimuthal wavenumber
m	mass
p	pressure
$\overline{p}$	$\int_{\theta_T}^{\theta}\frac{\partial\overline{p}}{\partial\theta}d\theta$ (See equation 3.6)
P_{00}	1000 hPa
S	streamfunction
t	time
z	height above sea level
$\hat{F}$	sum of torques (see equation 2.3)
J_θ	Jacobian for transformation, $\lvert\partial z/\partial\theta\rvert$
K	Kelvin
R	Gas constant for dry air
R_β	$-(a\sin\beta)^3$
T	temperature
Δp	amplitude of azimuthal wave
α	azimuthal coordinate
α^{*}	$(R/\overline{p})(\overline{p}/p_{oo})^{\kappa}$
β	radial coordinate defined by angular distance
θ	potential tempeature
ψ	Montgomery streamfunction
κ	R/c_p
ρ	Density
φ	latitude
φ_0	latitude of the center of a storm
Φ	radial coordinate in $(\Phi,\overline{p})$ coordinates defined by length
ω_p	$d\overline{p}/dt$

Subscripts and Superscripts

a	absolute frame of reference
B	bottom or boundary
o	budget volume central reference axis
$\overline{p}$	$\overline{p}$ coordinate
s	surface of the earth
T	top of atmosphere
z	vertical direction
α	azimuthal component
β	raidal component
θ	isentropic coordinate system; isentropic surface

Vectors

$\vec{F}$	frictional force
$\vec{U}$	earth-relative wind velocity
$\vec{W}$	earth-relative velocity of lateral boundary of a storm volume
$\vec{g}_a$	absolute angular momentum
$\vec{m}$	unit vector in radial direction
$\vec{l}$	unit vector in azimuthal direction
$\vec{k}$	unit vector in vertical direction
$\vec{r}$	position vector
$\vec{\Omega}$	angular velocity of the earth

Operators

$\frac{\partial}{\partial\theta}$	partial derivative with respect to θ
$\frac{\delta}{\delta t_\theta}$	quasi-Lagrangian time derivative in isentropic coordinates
$\frac{\partial}{\partial\Phi}$	$\frac{1}{a}\frac{\partial}{\partial\beta\overline{p}}$
$\cdot$	$\frac{d}{dt}$
$-$	azimuthal average
$\wedge$ or	< > mass-weighted azimuthal average
$*$	deviation from mass-weighted azimuthal average
$'$	deviation from azimuthal average
∇	horizontal del operator
$\int$	Riemann integral
$\lvert\ \rvert$	absolute value

APPEXDIX E

An Approximation of $\overline{p^\kappa}$ within the Tropospheric Structure of Cyclones

The power series expansion of $\overline{p^\kappa}$ may be expressed (Gallimore and Johnson, 1981) as

$$\overline{p^\kappa} = \overline{p}^\kappa [1 + \sum_{n=2}^{\infty} \frac{\kappa(\kappa-1)\cdots(\kappa-n+1)}{n!} \overline{(\frac{p'}{p})^n} = \overline{p}^\kappa [1 + s(p' / \overline{p})],$$

where the series $s(p' / \overline{p})$ is convergent. With k_n as an azimuthal wavenumber, the azimuthal pressure deviation p' is expressed by

$$p' = \Delta p \sin(k_n \alpha) \qquad 0 \leqslant \alpha \leqslant 2\pi.$$

If the ratio of Δp to $\overline{p}$ equals one to two, then $\overline{p^\kappa}$ is bounded (Gallimore and Johnson, 1981) by

$$0.9829\overline{p}^\kappa \leqslant \overline{p^\kappa} \leqslant \overline{p}^\kappa .$$

Therefore $\overline{p}^\kappa$ is likely a good approximation for $\overline{p^\kappa}$ within the tropospheric structure of cyclones. The smaller the p' (or Δp) is, the better the approximation will be.

REFERENCES

Challa, M. and R. L. Pfeffere (1980), Effects of eddy fluxes of angular momentum on model hurricane development, *J. Atmos. Sci.*, **37**: 1603–1618.

Czarnetzki A.C. and D.R. Johnson (1995), The role of terrain and pressure stresses in Rocky Mountain lee cyclones, *Mon. Wea. Rev.*, **124**: 553–570.

Dotton, J.A. (1976), *The ceaseless wind*, McGraw–Hill, 579pp.

Eliassen, A. (1951), Slow thermally or frictionally controlled meridional circulation in a circular vortex, *Astrophysica Norvebica*, **5**: 19–60.

Gallimore, R.G. and D.R. Johnson (1981), The forcing of the meridional circulation of the isentropic zonally averaged circumpolar vortex, *J.Atmos. Sci.*, **38**: 583–599.

Hale, R. (1983), Mass and angular momentum diagnostics of the intense Ohio Valley extratropical cyclone of 25–27 January 1978, M.S. thesis, University of Wisconsin–Madison, 96pp.

Johnson, D.R. (1974), The absolute angular momentum of cyclones, Subsynoptic Extratropical Weather System: *Observations, Analysis, Modeling, and Prediction, Volumn II seminars and workshops, Colloquium notes of the advanced study program and small–scale analysis and prediciton project*, National Center for Atmospheric Research, Boulder, CO, (NTIS PB–2472286). 821 pp.

----, and W.K. Downey (1975a), Azimuthally averaged transport and budget equations for storms: quasi-Lagrangian diagnostics 1, *Mon. Wea. Rev.*, **103**: 967–979.

----,and --- (1975b), The absolute angular momentum of storms: quasi–Lagrangian diagnostics 2, *Mon. Wea. Rev.*, **103**: 1063–1077.

---, and --- (1976), The absolute angular momentum budget of an extratropical cyclone: quasi–Lagrangian diagnostics 3, *Mon. Wea. Rev.*, **104**: 3–14.

---, --- and C.H. Wash (1981), The forcing of the extratropical cyclone within an angular momentum perspective, (unpublished manuscript).

---, and D.K. Hill (1987), Quasi-Lagrangian diagnostics of a Mediterranean cyclone: Isentropic results, *Meteorol. Atmos. Phys.*, **36**: 118–140.

Katzfey, J.J. (1978), A diagnostic study of the vertical redistribution of angular momentum in isentropic coordinates by pressure torques, M.S. thesis, University of Wisconsin–Madison, 118 pp.

Katzfey, J.J. (1983), On the role of the baroclinic structure in extratropical cyclone development, Ph.D. thesis, University of Wisconsin-Madison, 265pp.

Kuo, H–L (1956), Forced and free meridional circulations in the atmosphere, *J. Atmos. Sci.*, **13**: 561–568.

Lettau, H.H. (1959), Wind profile, surface stress and geostrophic drage coefficients in the atmospheric surface layer, *Advances in Geophysics*, Academic Press, **6**: 241–257.

Lorenz, E.N. (1955), Generation of available potential energy and the intensity of the general circulation. In "Large

Scale Synoptic Processes" (J. Bjerknes, proj. dir.), Final Rep., 1957, Contract AF 19 (604)–1286 Dept. Meteorol. Univ. of California, Los Angeles.

Molinari, M. and D. Vollaro (1990), External influences on hurricane intesity, Part II: Vertical structure and response of the hurricane vortex, *J. Atmos. Sci.*, **47**: 1902–1918.

Molinari, M., D. Vollaro and S. Skubis (1993), Application of the Eliassen balanced model to real–data tropical cyclones, *Mon. Wea. Rev.*, **121**: 2409–2419.

Pfeffer, R.L. and M. Challa (1981), A numerical study of the role of eddy fluxes of momentum in the development of Atlantic hurricanes, *J. Atmos. Sci*, **38**: 2393–2398.

Schneider, R. (1986), Quasi–Lagrangian diagnostics of the 9–14 April 1979 Great Plains extratropical cyclone and subsynoptic scales, M.S. thesis, University of Wisconsin–Madison, 152 pp.

Suhubert, W.H. and B.T. Alworth (1987), Evolution of potential vorticity in tropical cyclones, *Q.J.R. Meterorol. Soc.*, **113**: 147–162.

Sundqvist, H. (1970), Numerical simulation of the development of tropical cyclones with a ten–level model, Part I. *Tellus*, **4**: 359–390.

Wash, C.H. (1978), Diagnostics of observed and numerically simulated extratropical cyclones, Ph.D. thesis, University of Wisconsin–Madison, 215 pp.

Yuan, Z.J. (1990), On the forcing of the meridional circulation within cyclones: an isentropic study, M.S. thesis, University of Wisconsin–Madison, 72pp.