Retrieval Single–Doppler Radar Wind with Variational Assimilation Method—Part I: Objective Selection of Functional Weighting Factors

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ABSTRACT

In variational problem, the selection of functional weighting factors (FWF) is one of the key points for discussing many relevant studies. To overcome arbitrariness and subjectivity of the empirical selecting methods used widely at present, this paper tries to put forward an optimal objective selecting method of FWF. The focus of the study is on the weighting factors optimal selection in the variation retrieval single–Doppler radar wind field with the simple adjoint models. Weighting factors in the meaning of minimal variance are calculated out with the matrix theory and the finite difference method of partial differential equation. Experiments show that the result is more objective comparing with the factors obtained with the empirical method.

Key words: Variation, Weighting factor, Minimum variance, Objective selection

I. INTRODUCTION

At present, one of the main ways for studying numerical weather forecast is to transfer the initial value problem of differential equation into the extremum problem of functional. Its basic idea is to take the dynamic constraint and data constraint as a whole. With the multi time levers observational data of a time period, the data evolution information with time is converted into the spatial distribution of the elements field with the help of the numerical models, as is also the basic principle of the four–dimensional data assimilation (FDDA) method. In the same way, at present the single–Doppler radar data variational assimilation retrieval technique has also gotten constant development (Kapitza, 1991; Sun et al., 1991, 1994; Qiu et al., 1992; Laroche et al., 1994; Xu et al., 1994, 1995). Observational data of various time levers are put into the various numerical models established with the basic control equations of atmospheric motion, converting the initial value problem of the differential equation into the extremum problem of functional, namely, variational problem. The various parameters of the models are retrieved, so as to obtain the real distribution of wind direction, velocity, temperature and air pressure in the rainfall system.

In variational problems, the selection of FWF is one of the key points for discussing many relevant studies. The selection will affect the result of numerical forecast and retrieval. At present, the selection of FWF mainly depends on the man's judgment on models and the confidence degree of various data. It is arbitrary and subjective to some degree. The main problems of the selection method are as follows:

(1) lacking the coordination with numerical models and observational data;

(2) lacking the coordination among the weighting factors;

(3) lacking objectivity, weighting factors are often determined by experience;

(4) the value has to be obtained with a large amount of calculation.

Thus, Chou Jifan pointed out(1995) that it is necessary to overcome these difficulties through mathematics and calculating mathematics studies. Some tentative research has been done in this aspect (Liu, 1995). The aim of this paper is to put forward an optimal objective selecting method of FWF. The focus of the study is on the weighting factor optimal selection in the variation problem of the retrieval single–Doppler radar wind field with the simple adjoint models. Weighting factors in the meaning of minimal variance are calculated out with the matrix theory and the finite difference method of partial differential equation. On the basis of theoretical derivation, this paper has conducted some numerical experiments with the observational data and checked the results with the dual–Doppler radar data. It is shown that the result obtained with the mathematical and calculating mathematical method is more feasible, more objective. Comparing with the factors obtained with the empirical method (Sun et al., 1994, Xu et al., 1995), its values are also consistent with that, and fall in the nearby of best range. The coordination between the weighting factors, numerical models and the observational data and one among the weighting factors themselves has been achieved. With its theoretical basis as well as its wide prospects of application, this objective selecting method is probably a way towards the finding of the optimal weighting factors in the variational problem. It is not only applicable to the retrieval wind field of single–Doppler radar with variational assimilation method, but also can be gradually extended to the numerical forecast of various models of various scales, as well as FDDA. The following will be devoted to the detailed discussion of the mathematical derivation and numerical experiment with this method. The second part introduces the mathematical method, and the third part discusses the results of experiment. The fourth part is the conclusion.

II. METHOD

In the research of the simple adjoint model retrieval wind field, the basic control equations include the momentum equation of radial velocity and the advective equation of radar reflectivity factors on Cartesian coordinates and cylinder coordinates, etc. We take the retrieval of the reflectivity factor advective equation on the Cartesian coordinates as an example to explain the basic idea and method of the optimal selecting of FWF, because it is not only applicable to the single–Doppler radar data but also to the conventional weather radar data. In those areas of fewer Doppler radars it also has wide application.

1. *The Problem*

Let the reflectivity factor η of rain intensity satisfy the Lagrangian conservation in the integrated period, its control equation is:

$$\frac{\partial \eta_e}{\partial t} + u\frac{\partial \eta_e}{\partial x} + v\frac{\partial \eta_e}{\partial y} = - w_{ob}\frac{\partial \eta_{ob}}{\partial z}, \tag{2.1}$$

in which, " t" stands for time, (x,y,z) for the spatial position of the Cartesian coordinates, (u,v) is the 2–dimension horizontal wind field waiting to be retrieved, the subscript "e" stands for the estimate value of the controlling variable, "ob" is the observational value.

To retrieve 2–dimension horizontal wind field (u, v), the cost function with the horizontal divergence as weak constraint has the following form:

$$J=\frac{1}{\Omega\tau}\int_0^\tau\iint_\Omega\left(w1(\eta_e-\eta_{oh})^2+w2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2\right)d\Omega dt, \tag{2.2}$$

where $w1$, $w2$ are weighting factors. w1 is called observation constraint, showing the error between the analysis field and the observation field; $w2$ is called kinematics constraint, controlling the horizontal divergence of the retrieval wind field.

In the present paper, functional with one constraint condition is studied first in order to discuss the optimal selection of FWF, which is helpful for the explanation of the basic method. Generally speaking, in different studies, cost functional with different constraints can be put forward according to actual requirements. Cost functional of these types all contains weighting factors needed to be determined first. By qualitative analysis of (2.2), it is known that the selection of larger $w1$ and smaller $w2$ means that the analysis field distribution is closer to that of observation field and on the contrary, the selection of smaller $W1$ and larger $w2$ means more emphasis on the horizontal divergence constraint (Sasaki, 1970). Furthermore, in some problems, that parameter error of equation causes solution error increases with time, so weighting factor $w1$ should decrease with time increasing (Chou, 1995). Thus we can see, different weighting factor selecting method has different effects on the solution in retrieval problem. The often used empirical and statistical method is subjective, lacking objective basis. Therefore, how to determine these weighting factors so as to make best the analysis field obtained with variational method is the very problem to be solved urgently. The present paper determines these weighting factors by using minimal variance principle under reasonable assumption.

2. *Mathematical Hypotheses*

Please refer to Appendix A for the definitions of the first–order, second–order difference operators and their exposition.

For the later deduction, according to the corresponding models of real problems, we make the following mathematical hypotheses:

H1: the model of observation field on point (i, j) is as follows:

$$\tilde{\varphi}_{i,j}=\varphi^t_{i,j}+n_{i,j}, \tag{2.3}$$

where $\varphi^t_{i,j}$ is ideal field, $n_{i,j}$ is noise with zero mean value, then

$$E(n_{i,j}\cdot\varphi^t_{i,j})=0, \quad \text{for all } (i,j) \text{ points} \tag{2.4}$$

$$\text{and } Var(n_{i,j})=\sigma_0^2, \quad \text{for all } (i,j) \text{ points} \tag{2.5}$$

σ_0^2 is variance of noise, obtained with filtering the mean of nine points observational data. Please refer to Appendix B for details.

$$\text{H2:} \quad E(\nabla_x\overline{\nabla}_x\varphi^t_{i,j})=0, \quad \text{for all } (i,j) \text{ points} \tag{2.6}$$

$$\text{and} \quad Var(\nabla_x\overline{\nabla}_x\varphi^t_{i,j})=\sigma_1^2, \quad \text{for all } (i,j) \text{ points} \tag{2.7}$$

σ_1^2 is the variance of second order difference operator in ideal field. For solution method see Appendix B.

H3: $Var(\varphi'_{i,j})=\sigma_2^2$, for all (i,j) points (2.8)

σ_2^2 is the variance of ideal field, see details of solution in Appendix B.

3. *Variational Problem and Euler Equation*

For variational problem $\delta J=0$, the Euler equation to attain is neat. So let $k=\frac{w2}{w1}$, then the variation in the cost function (2.2) is

$$\delta J=\frac{1}{\Omega\tau}\int_0^{\tau}\iint_{\Omega}\left(2(\eta_e-\eta_{ob})\delta\eta+2k\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\delta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right)d\Omega dt, \tag{2.9}$$

where $\delta\eta$ is the variation of η_e.

For (2.1), to get first order variation, we have the following form:

$$\frac{\partial\delta\eta}{\partial t}+u\frac{\partial\delta\eta}{\partial x}+v\frac{\partial\delta\eta}{\partial y}=-\delta u\frac{\partial\eta_e}{\partial x}-\delta v\frac{\partial\eta_e}{\partial y}. \tag{2.10}$$

The left part of the equation can be changed as: $\frac{d\delta\eta}{dt}$, (2.10) is integrated from time $0\rightarrow\tau$, and we obtain

$$\delta\eta=-\tau\left(\delta u\frac{\partial\eta_e}{\partial x}+\delta v\frac{\partial\eta_e}{\partial y}\right) \tag{2.11}$$

Substituting the variational problem $\delta J=0$ of (2.9) and expanding the various items and by integration with the boundary conditions $u|_{\Gamma}=0$, $v|_{\Gamma}=0$, the following Euler system of equations is obtained:

$$\begin{aligned}&\tau\frac{\partial\eta_e}{\partial x}(\eta-\eta_{ob})+k\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}\right)=0,\\&\tau\frac{\partial\eta_e}{\partial y}(\eta-\eta_{ob})+k\left(\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial y^2}\right)=0.\end{aligned} \tag{2.12}$$

In mathematics, the solution of problem (2.12) exists and is unique answer, that is, the analysis field can uniquely be obtained from the observation field. Here η is the model variable and be obtained with integrating control equation, u,v are the parameters waiting to be retrieved in model.

Plus the two equations and setting

$$\mu=\frac{k}{\tau\left(\frac{\partial\eta_e}{\partial x}+\frac{\partial\eta_e}{\partial y}\right)} \tag{2.13}$$

then the Euler equation is

$$\eta_e+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 u}{\partial y\partial x}+\frac{\partial^2 v}{\partial y^2}\right)=\eta_{ob}. \tag{2.14}$$

In actual problem, observation field is often discreted, so (2.14) must be discreted.

4. *Difference Format*

Put (2.14) into the discrete difference equation with the seven points difference format:

$$\eta_{i,j}+\mu(\nabla_x\overline{\nabla}_x u_{i,j}+\nabla_x\overline{\nabla}_y v_{i,j}+\nabla_y\overline{\nabla}_x u_{i,j}+\nabla_y\overline{\nabla}_y v_{i,j})=\eta_{obi,j}, \qquad (2.15)$$

where "i" stands for row, "j" for column ($i=1,2,\cdots,m-1$; $j=1,2,\cdots n-1$). Its boundary condition is $s_{i,0}=s_{o,j}=s_{i,n}=s_{m,j}=0$. "$s$" separately stands for η, u, v, where $i=0,1,\cdots,m$; $j=0,1,\cdots,n$.

Let $a=\frac{\mu}{h^2}$, h is grid length, write out the (2.15) in terms of grid point:

$$\eta_{i,j}-3au_{i,j}+au_{i+1,j}+au_{i,j-1}+2au_{i-1,j}-au_{i-1,j+1}-3av_{i,j}+av_{i+1,j}+av_{i,j+1}$$
$$+2av_{i,j-1}-av_{i+1,j-1}=\eta_{obi,j}. \qquad (2.16)$$

5. *Matrix Form of the Difference Equation*

Now we turn to the solution problem (2.16),

Let:

analysis field $\overline{F}=\begin{bmatrix}\overline{\eta}\\ \overline{U}\\ \overline{V}\end{bmatrix}$, observation field $\overline{F}^*=\begin{bmatrix}\overline{\eta}^*\\ \overline{U}^*\\ \overline{V}^*\end{bmatrix}$, ideal field $\overline{F}^t=\begin{bmatrix}\overline{\eta}^t\\ \overline{U}^t\\ \overline{V}^t\end{bmatrix}$.

where,

$$\overline{\eta}=\left[\eta_{11},\eta_{12},\cdots,\eta_{1,n-1},\eta_{21},\cdots,\eta_{2,n-1},\cdots,\eta_{m-1,1},\cdots,\eta_{m-1,n-1}\right]^T,$$
$$\overline{\eta}^*=\left[\eta^*_{11},\eta^*_{12},\cdots,\eta^*_{1,n-1},\eta^*_{21},\cdots,\eta^*_{2,n-1},\cdots,\eta^*_{m-1,1},\cdots,\eta^*_{m-1,n-1}\right]^T,$$
$$\overline{U}=\left[u_{11},u_{12},\cdots,u_{1,n-1},u_{21},\cdots,u_{2,n-1},\cdots,u_{m-1,1},\cdots,u_{m-1,n-1}\right]^T,$$
$$\overline{V}=\left[v_{11},v_{12},\cdots,v_{1,n-1},v_{21},\cdots,v_{2,n-1},\cdots,v_{m-1,1},\cdots,v_{m-1,n-1}\right]^T,$$

then the difference equation (2.16) can be expressed with matrix form as

$$\tilde{A}\overline{F}=\overline{F}^*, \qquad (2.17)$$

where $\tilde{A}$ is the block diagonal matrix, $\tilde{A}=\begin{bmatrix}\tilde{A}_1 & & \\ & \tilde{A}_2 & \\ & & \tilde{A}_3\end{bmatrix}$

then $$\begin{bmatrix}\tilde{A}_1 & & \\ & \tilde{A}_2 & \\ & & \tilde{A}_3\end{bmatrix}\begin{bmatrix}\overline{\eta}\\ \overline{U}\\ \overline{V}\end{bmatrix}=\begin{bmatrix}\overline{\eta}^*\\ 0\\ 0\end{bmatrix}. \qquad (2.18)$$

whereas $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ are respectively corresponding with $\overline{\eta},\overline{U},\overline{V}$, partial matrix. They are all the square matrix of $[(m-1)(n-1)]\times[(m-1)(n-1)]$, having respectively the following forms:

$$\tilde{A}_1 = \begin{bmatrix} \tilde{I} & & & & \\ & \tilde{I} & & & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & \tilde{I} \end{bmatrix}$$

$$\tilde{A}_2 = \begin{bmatrix} \tilde{D}_1 & \tilde{D}_2 & & & \\ \tilde{D}_3 & \tilde{D}_1 & \tilde{D}_2 & & \\ & \tilde{D}_3 & \cdots & \cdots & \\ & & \cdots & \cdots & \tilde{D}_2 \\ & & & \tilde{D}_3 & \tilde{D}_1 \end{bmatrix} \quad \tilde{A}_3 = \begin{bmatrix} \tilde{D}_4 & \tilde{D}_5 & & & \\ & \tilde{D}_4 & \tilde{D}_5 & & \\ & & \cdots & \cdots & \\ & & & \cdots & \tilde{D}_5 \\ & & & & \tilde{D}_4 \end{bmatrix}$$

where $\tilde{I}$ is the unit matrix of $(m-1)\times(m-1)$. $\tilde{D}_1, \tilde{D}_2, \tilde{D}_3, \tilde{D}_4, \tilde{D}_5$ are the square matrix of $(m-1)\times(m-1)$.

$$\tilde{D}_1 = \begin{bmatrix} -3a & a & & & \\ & -3a & a & & \\ & & \cdots & \cdots & \\ & & & \cdots & a \\ & & & & -3a \end{bmatrix} \quad \tilde{D}_2 = \begin{bmatrix} a & & & & \\ & a & & & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & a \end{bmatrix}$$

$$\tilde{D}_3 = \begin{bmatrix} 2a & -a & & & \\ & 2a & -a & & \\ & & \cdots & \cdots & \\ & & & \cdots & -a \\ & & & & 2a \end{bmatrix} \quad \tilde{D}_4 = \begin{bmatrix} -3a & a & & & \\ 2a & -3a & a & & \\ & 2a & \cdots & \cdots & \\ & & \cdots & \cdots & a \\ & & & 2a & -3a \end{bmatrix}$$

$$\tilde{D}_5 = \begin{bmatrix} a & & & & \\ -a & a & & & \\ & -a & \cdots & & \\ & & \cdots & \cdots & \\ & & & -a & a \end{bmatrix}$$

6. *Several Lemmas*

Before discussion, we first introduce several known lemmas.

Lemma 2.1 If $\tilde{A} = \left[a_{i,j} \right]$ is a strict diagonal dominant matrix, then $\tilde{A}$ is nonsingularity matrix.

Lemma 2.2 If $\tilde{A}$ is nonsingularity matrix, and the proportion of time and space steps satisfies the stability condition $\left| C\frac{\Delta t}{\Delta x} \right| \leqslant 1$, then the solution of equation (2.18) exists and is unique, i.e.

$$\tilde{P} = \tilde{A}^{-1}\tilde{P}^{*}. \tag{2.19}$$

Lemma 2.3 If $\tilde{A} = \left[a_{i,j}\right]$ is a matrix of m × n, then

$$Tr\tilde{A}^T\tilde{A} = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{i,j}^2,$$

here "Tr" stands for the trace of the matrix.

Lemma 2.4 If the $\tilde{A}_i (i = 1,2,\cdots,s)$ of the block diagonal matrix $\tilde{A}$ are all inverse matrix, then

$$\tilde{A}^{-1} = \begin{bmatrix} \tilde{A}_1 & & & \\ & \tilde{A}_2 & & \\ & & \cdots & \\ & & & \tilde{A}_s \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{A}_1^{-1} & & & \\ & \tilde{A}_2^{-1} & & \\ & & \cdots & \\ & & & \tilde{A}_s^{-1} \end{bmatrix}$$

7. *The Variance of the Analysis Field and the Ideal Field*

To describe the closeness of the analysis field $\tilde{P}$ and the ideal field $\tilde{P}^{t}$ to be solved in variational analysis, take the variance $Var(\tilde{P} - \tilde{P}^{t})$ between $\tilde{P}$ and $\tilde{P}^{t}$ into consideration.

$$\begin{aligned} Var(\tilde{P} - \tilde{P}^{t}) &= Var(\tilde{A}^{-1}\tilde{P}^{*} - \tilde{P}^{t}) \\ &= [\tilde{A}^{-1}]Var(\tilde{P}^{*} - \tilde{A}\tilde{P}^{t})[\tilde{A}^{-1}]^T \\ &= [\tilde{A}^{-1}]Var[(\tilde{P}^{*} - \tilde{P}^{t}) - [\tilde{A} - \tilde{I}]\tilde{P}^{t}][\tilde{A}^{-1}]^T, \end{aligned} \tag{2.20}$$

where

$$Var(\tilde{P}^{*} - \tilde{P}^{t}) = \sigma_0^2, \tag{2.21}$$

and because

$$\left[\tilde{A} - \tilde{I}\right] = \begin{bmatrix} \tilde{A}_1 & & \\ & \tilde{A}_2 & \\ & & \tilde{A}_3 \end{bmatrix} - \begin{bmatrix} \tilde{I} & & \\ & \tilde{I} & \\ & & \tilde{I} \end{bmatrix} = \begin{bmatrix} \tilde{A}_1 - I & & \\ & \tilde{A}_2 - \tilde{I} & \\ & & \tilde{A}_3 - \tilde{I} \end{bmatrix},$$

expanding each term: $\left[\tilde{A}_1 - \tilde{I}\right] = 0$

$$\left[\tilde{A}_2 - \tilde{I}\right] = \begin{bmatrix} \tilde{D}_1 - \tilde{I} & \tilde{D}_2 & & & \\ \tilde{D}_3 & \tilde{D}_1 - \tilde{I} & \tilde{D}_2 & & \\ & \tilde{D}_3 & \cdots & \cdots & \\ & & \cdots & \cdots & \tilde{D}_2 \\ & & & \tilde{D}_3 & \tilde{D}_1 - \tilde{I} \end{bmatrix}$$

$$\left[\tilde{D}_1-\tilde{I}\right]=\begin{bmatrix} -3a-1 & a & & & \\ & -3a-1 & a & & \\ & & \cdots & \cdots & \\ & & & \cdots & a \\ & & & & -3a-1 \end{bmatrix}$$

$$\left[\tilde{A}_3-\tilde{I}\right]=\begin{bmatrix} \tilde{D}_4-\tilde{I} & \tilde{D}_5 & & & \\ & \tilde{D}_4-\tilde{I} & \tilde{D}_5 & & \\ & & \cdots & \cdots & \\ & & & \cdots & \tilde{D}_5 \\ & & & & \tilde{D}_4-\tilde{I} \end{bmatrix}$$

$$\left[\tilde{D}_4-\tilde{I}\right]=\begin{bmatrix} -3a-1 & a & & & \\ 2a & -3a-1 & a & & \\ & 2a & \cdots & \cdots & \\ & & \cdots & \cdots & a \\ & & & 2a & -3a-1 \end{bmatrix}.$$

Make $[\tilde{A}-\tilde{I}]P^{\tau}$ into difference format:

$$[\tilde{A}-\tilde{I}]P^{\tau}=(\mu(\nabla_x\overline{\nabla}_x u'_{i,j}+\nabla_x\overline{\nabla}_y v'_{i,j}+\nabla_y\overline{\nabla}_x u'_{i,j}+\nabla_y\overline{\nabla}_y v'_{i,j})-u'_{i,j}-v'_{i,j}). \quad (2.22)$$

thus,

$$\begin{aligned} &Var([\tilde{A}-\tilde{I}]P^{\tau})\\ &=Var[\mu(\nabla_x\overline{\nabla}_x u'_{i,j}+\nabla_x\overline{\nabla}_y v'_{i,j}+\nabla_y\overline{\nabla}_x u'_{i,j}+\nabla_y\overline{\nabla}_y v'_{i,j})-u'_{i,j}-v'_{i,j}], \end{aligned} \quad (2.23)$$

From the previous reasonable assumption, we obtain:

$$Var(\bar{P}-P^{\tau})=[\tilde{A}^{-1}][\sigma_0{}^2+\mu^2(\sigma_{11}^2+\sigma_{12}^2+\sigma_{13}^2+\sigma_{14}^2)-\sigma_{21}^2-\sigma_{22}^2][\tilde{A}^{-1}]^T \quad (2.24)$$

in which σ_{1i}^2 (i= 1,2,3,4) correspond respectively to the variance of the second order difference operators in the four different ideal fields, σ_{2i}^2 (i= 1,2) correspond respectively to the variance of the two ideal fields.

8. *The Trace of the Variance*

Because of in variance matrix $Var(\bar{P}-P^{\tau})$ of (2.24), each element on the main diagonal line is the variance for each of the components of estimate error, so we can select the parameter μ to make the sum of all the elements (variance sum) on the main diagonal line of $Var(\bar{P}-P^{\tau})$ be the minimal (Chui, et al., 1991). Thus, get the trace of (2.24): $TrVar(\bar{P}-P^{\tau})$, and the parameter μ can be obtained from the following equation:

$$\frac{\partial}{\partial\mu}TrVar(\bar{P}-P^{\tau})=0. \quad (2.25)$$

Put parameter μ back into (2.13), where the integrated time period τ is known, and $\left(\frac{\partial \eta_e}{\partial x}+\frac{\partial \eta_e}{\partial y}\right)$ cannot be known in advance, it can be put into the cost function J and be calculated dynamically. Based on this, the weighting factors w_1 and w_2 of the cost function can be selected according to their ratio value.

Through the above deduction, we can know theoretically that optimal selection exists in the weighting factors of the cost function under the minimal variance of analysis and ideal field in terms of the difference method of matrix theory and partial differential equation, if the Euler equation of the constraint variational problem is discreted into difference format and satisfies the stableness condition of the difference equation and the reasonable assumption from the actual problem.

III. EXPERIMENT RESULT DISCUSSION

The deduction of the optimal selection of FWF has been finished theoretically so far, the next is to test its application and its effect in the model. For this purpose, the experiments in the following aspects have been designed:

(1) To study the key problems in the matrix calculation with this method and the possibility of programming;

(2) To make numerical analysis of those terms to be dynamically calculated in the model with the dual–Doppler radar data;

(3) To make comparison between this objective method and the general empirical method;

(4) To discuss the improvement of retrieval result with the objective selecting.

To check the feasibility of the above method, the present paper makes numerical experiments with the observed Doppler radar rainfall data, and makes comparison by using the corresponding dual–Doppler radar synthetic wind field as the "real value". The data used here are about one rather strong meiyu rainfall collected from NAO and WAGU Stations in Japan on July 2, 1993. Both the two stations are X–band Doppler radar. Please refer to next paper (part II) about the details of simple adjoint model and the real data. The related results of experiment is mainly discussed here.

1. *Selection of Difference Format*

When calculating FWF, the seven points difference format in (2.15) is used in the Euler equation. In the difference method of partial differential equation, the selection of proper difference format will directly influence calculation result, because the coefficient matrix A of these difference equations has the features of large sparseness, diagonally dominance and irreducibility. From Lemma (2.1) and Lemma (2.2) we know that A can only be assured to be nonsingular matrix with $\tilde{A}$'s diagonal dominance so as to make the difference equation have only one solution. Obviously, the selection of difference format must meet the requirement of diagonal dominance, i.e. Matrix $\tilde{A}=[a_{ij}]$ satisfy the inequality $|a_{ii}|\geqslant \sum_{j\neq i}|a_{ij}|$, and there must be at least one row that makes the inequality strictly be established. The difference format selected by this paper satisfies the diagonal dominance.

According to $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$, this paper also tested another difference format $\nabla_x \overline{\nabla}_y u_{ij}$ which is different from the $\nabla_y \overline{\nabla}_x u_{ij}$ in (2.15) so as to ensure that the different forms of difference operators will not influence the result. Take the block matrix $\tilde{A}_2$ as an example, in deduction, we found that the form of Matrix $\tilde{D}_1$ remains unchanged, while $\tilde{D}_2$ and $\tilde{D}_3$ change into the form that is symmetrical to the original, namely,

$$\tilde{D}_2 = \begin{bmatrix} 2a & & & & \\ -a & 2a & & & \\ & -a & \cdots & & \\ & & \cdots & \cdots & \\ & & & -a & 2a \end{bmatrix}, \quad \tilde{D}_3 = \begin{bmatrix} a & & & & \\ & a & & & \\ & & \cdots & & \\ & & & \cdots & \\ & & & & a \end{bmatrix}$$

Further calculation shows that $\tilde{A}_2^{-1}$ has also a symmetrical form with the original $\tilde{A}_2^{-1}$. In consequence, the result of variance trace is in accordance with the original. The same is true with $\tilde{A}_3$, $\tilde{A}_3^{-1}$, which indicates that symmetrical matrix has the same trace of inverse variance. Thus, we can conclude that so far as the second-order derivation $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ is concerned, although the different selection of difference format will make the matrix form different, they are symmetrical to each other and their trace of variance is the same and their calculation result is completely the same.

2. *Calculation of Coefficient Matrix and Trace*

The selection of functional weighting factors with this method chiefly lies in the various calculations of the coefficient matrix of difference equation. Many experiments have been done again and again to achieve the dependable and stable computer treatment.

Matrix $\tilde{A}$ in (2.18) is a large block diagonal matrix, where elements $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{A}_3$ are also block matrix respectively. The amount of elements depends on the amount of data grid points. When the amount of grid points in X direction is $i = 0,1,\cdots m, m = 21$; and in Y direction is $j = 0,1,\cdots n, n = 21$, $Ai(i = 1,2,3)$ is the square matrix of $(m-1)(n-1) \times (m-1)(n-1)$, i.e. 361×361 square matrix. And matrix A is a large sparse matrix of 1083×1083.

In calculation of A's inverse matrix $\tilde{A}^{-1}$, according to Lemma 2.4, the inverse matrixes $\tilde{A}_1^{-1}$, $\tilde{A}_2^{-1}$, $\tilde{A}_3^{-1}$ of each element are calculated out respectively. Standard calculating programming is applied in the calculation of inverse matrixes and its dependability is proved through once more inverse operation.

In calculation of (2.25), first the trace of variance in the analysis field and ideal field has to be worked out. In the same, according to Lemma 2.4, the calculation of trace can also be simplified, i.e.

$$Tr(\tilde{A}^{-1})(\tilde{A}^{-1}) = \sum_{i=1}^{3} Tr(\tilde{A}_i^{-1})(\tilde{A}_i^{-1})^T.$$

Then according to different forms of the calculation result of $TrVar(\tilde{P} - P^t)$, either analytic or numerical iteration form is used to get solution. In the experiment of this paper, the calculation result of $\frac{\partial}{\partial \mu} TrVar(\tilde{P} - P^t) = 0$ is unary quartic equation, so the μ's analytic value can be obtained. If the result of (2.25) is not the simple analytic, then numerical iteration method is used to get solution, making parameter μ increase (decrease) in the positive (negative) numerical direction starting from 0 with very small step length (for instance, 0.001), do it for several thousand times to find the parameter μ when $TrVar(\bar{P}-\bar{P}^{\tau})\mapsto \min$. The experiment has also used the numerical iteration method to get μ, its result is consistent with the analytic value with the error within 0.001.

All the above calculations can be done quickly and conveniently with mathematics programming. In a word, the form of the coefficient matrix can be determined after the selection of a certain functional. The change happens only in the calculation of the various variances in (2.24) and the dynamic calculation of $\left(\frac{\partial \eta_e}{\partial x}+\frac{\partial \eta_e}{\partial y}\right)$ which will be discussed in the next step.

3. *The Calculation of* $\sigma_0^2, \sigma_1^2, \sigma_2^2$ *and* $\left(\frac{\partial \eta_e}{\partial x}+\frac{\partial \eta_e}{\partial y}\right)$

The calculation method of various variances about $\sigma_0^2, \sigma_1^2, \sigma_2^2$ is demonstrated in Appendix B. However, its completion in the programming needs to be further discussed, because the various variances of the wind field cannot be known in advance. To obtain their value range, synthetic wind field data of dual–Doppler radar are used in the experiment of this paper to calculate $\sigma_0^2, \sigma_{1i}^2 (i=1,2,3,4)$ and $\sigma_{2i}^2 (i=1,2)$ in (2.24). For study and experiment, the usage of dual–Doppler radar data is dependable and stable. The example is the data obtained at 15:45 on July, 2, 1993. The calculation results at 0.5 km height are $\sigma_0^2=5.7$, $\sigma_{11}^2=0.15$, $\sigma_{12}^2=0.05$, $\sigma_{13}^2=0.05$, $\sigma_{14}^2=0.05$, $\sigma_{21}^2=0.63$, $\sigma_{22}^2=0.54$. In the future operational practice of retrieval with single–Doppler radar data, the calculation of various variances can be obtained dynamically in the numerical model. If it is not retrieval parameters but model variables, calculation will be comparatively simple. The results can be directly obtained from the mean observational data. A method has already been worked out to solve these variances dynamically by using (u, v) fields constantly produced in the process of model iteration. It will be left to be discussed in later study.

Because η_e is not a retrieval parameter, $\left(\frac{\partial \eta_e}{\partial x}+\frac{\partial \eta_e}{\partial y}\right)$ can be put into the cost functional and calculated dynamically in the process of model iteration. To obtain the value range, calculation is done by using the time and space mean observation data $\bar{\eta}_{ob}$ of single–Doppler radar and obtain $\left(\frac{\partial \bar{\eta}_{ob}}{\partial x}+\frac{\partial \bar{\eta}_{ob}}{\partial y}\right)=1.945$.

The above data obtained from the dual–Doppler radar and observation have provided effective information for the estimation of numerical values of FWF so as to further discuss the similarities and differences between this result and empirical value.

4. *Comparison between Objective Selection and Empirical Selection*

The selecting method for FWF deduced in this paper is an objective method. All weighting factors are put into Euler equation of variational problem. Select parameter μ under the minimal variance of analysis field and ideal field on the basis of difference method of partial differential equation and matrix theory, making the sum of each element on the main diagonal line (variance sum) in $Var(\bar{P}-\bar{P}^{\tau})$ matrix being minimal. The selected parameter μ has already had the basic property of a weighting factor. On this basis, the weighting factors dynamically selected are in good coordination with the numerical model and observational data, because the cost functional includes both information on numerical model and on observational data. Similarly, this method also has good coordination among weight factors themselves. The selection is not done one by one seperately. It is the calculation in the whole coefficient matrix $\tilde{A}$, which is good for objective analysis of the role and the order of magnitude of each term in the functional. Therefore, it avoids the subjective arbitrariness. The selection can be done steadily and dependably by computer program.

The next is the comparison of the features from the point of weighting factor value–taking in both methods.

To compare the numerical values, the above mentioned calculation result of dual–Doppler radar data and time and space mean observational data is adopted and we obtain $\frac{w2}{w1} = 2054$, i.e. when $w1 = 1$, $w2 = 2054$, which indicates that the weight factors of divergence constraints are pretty large.

Then let's study the selected weighting factors with empirical method done by other authors previously. Some papers simply took the constraints of functional simply as 1, 0.1 or 0.001 (i.e. Sun et al. 1994); some papers selected some function form. For example, Xu et al. (1994, 1995) put w1 into the form of decrease with time: $w1 = \left(\frac{\tau}{t + \Delta t}\right)^{\frac{1}{2}}$ where τ is integration time period, t is time, Δt is integration time step. And $w2 = k\sigma_\eta^2 w1_m$, $w1_m = \frac{1}{\tau}\int_0^\tau w1 dt$.

θ_η is the root–mean–square of spatial variation of the initial reflectivity field η_{ob}. Experiment is done to this empirical selection method with the data used in this paper, we obtain $w1_m = 1.56$, $A\sigma_\eta^2 = 3.83$. According to the author's description, k value can be taken between 50–1000, so the corresponding $w2$ is between 299–5975, which causes a large variation range for weighting factors, leading to indefiniteness. By comparison we know that the numerical value selected with objective method is within the empirical selecting range. It is not a random selection in the pretty large range. Meanwhile, by analysis, we know that the divergence constraint $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2$ is smaller than the numerical value of $(\eta_e - \eta_{ob})^2$ by 10^3. To show the role of each term in the functional balancedly, $w2$ will have a pretty large value, and their numerical values are also consistent. Besides, weighting factors selected with the objective method need to be calculated dynamically against the model, therefore, they have the feature of changing against time and space.

From the above analysis, we know that the two methods are not contradictory. The values obtained via objective method are consistent with those from empirical method, and more reasonable.

5. *Retrieval Result Analysis*

What we are most concerned about in the experiment is the role of weighting factor's optimal selection on the improvement of retrieval. It is a key whether this objective method can improve the wind retrieval.

To check the closeness between the retrieval wind field and the synthetic wind field of dual–Doppler radar, formula has been introduced in this paper to conduct error analysis. Supposing the grid points in X direction is m and in Y direction is n, the total grid points are $N = m \times n$. For any vector function (a,b), there is $\|(a,b)\|^2 \equiv \langle a,a \rangle + \langle b,b \rangle$. Then MVE (mean vector error) formula is as follows:

$$MVE = \frac{1}{N}\sum_{i=1}^{N} |(u_{ei} - u_{obi}, v_{ei} - v_{obi})|$$
$$= \frac{1}{N}\sum_{i=1}^{N} \sqrt{(u_{ei} - u_{obi})^2 + (v_{ei} - v_{obi})^2},$$

where the subscript "e" is retrieved wind, "ob" stands for synthetic wind of dual-Doppler radar.

In experiment, let $w1 = 1$, range of $w2$ is from 200 to 6000. MVE is analyzed to the retrieval wind and dual-Doppler radar synthetic wind. Result is shown in Fig. 1.

We know from Fig. 1 that MVE varies with $w2$. When nearby $w2 = 2054$, the error of retrieval is small. We also analyze the other heights data. Calculation results are: $w2 = 1995$ at 1 km height and $w2 = 1937$ at 1.5 km height. Their changes of MVE with $w2$ are similar to Fig. 1. Please refer to Fig. 2 and Fig. 3.

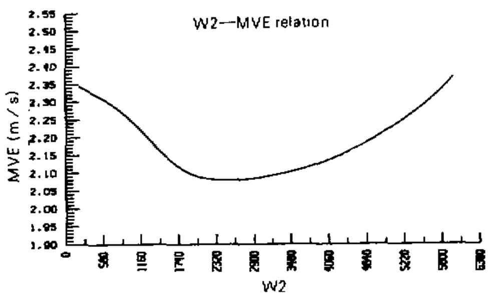

Fig.1. The relation of $w2$ and MVE at 0.5 km height, horizontal axis is $w2$, vertical axis is MVE.

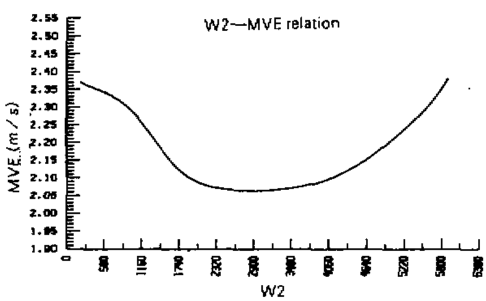

Fig.2. The relation of $w2$ and MVE at 1 km height, horizontal axis is $w2$, vertical axis is MVE.

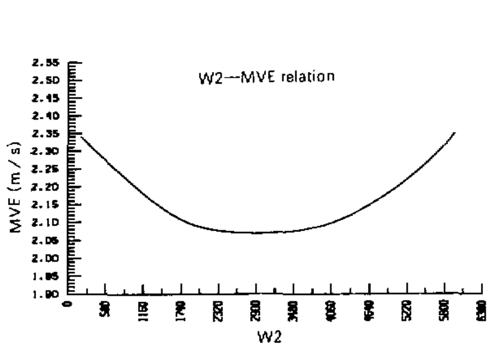

Fig.3. The relation of $w2$ and MVE at 1.5 km height, horizontal axis is $w2$, vertical axis is MVE.

Furthermore, we also see from above figures that there is a wide range of $w2$ corresponding to the smaller MVE. It is known by analysis that this due to the nonlinearity of dynamics model for retrieval wind field, namely, $w2$ is not a simple linearity relation to retrieval wind (u,v). It is also a basic reason about the large subjective arbitrariness and wider numerical range with empirical selecting method.

It is known from above analyses that comparing subjective selection methods, this method can get a better retrieval and decrease the number of experiments. It takes $w2$ within the good range and makes retrieval wind closer to real wind. Please refer to next paper (part II) about further discussion of retrieval results with simple adjoint model.

IV. CONCLUSION

The various parameters of variational assimilation method of single–Doppler radar data including retrieval wind, temperature and pressure fields are being studied and further developed, among which the selection of its FWF is one of the main concerns for scholars. At present, the selection of weighting factors in variational problem mainly depends on subjective judgement of the confidence in models and different data. Thus, the selection is subjective to some degree and lacks good coordination as a whole. Therefore, it is urgently necessary to conduct mathematics and calculation mathematics research in order to overcome the above mentioned shortcomings. The present paper makes a tentative deduction in mathematics and puts forward an objective method of optimal selection of weighting factors. Through numerical experiment with observational data and checking with dual–Doppler radar data, the method is proved satisfying. Conclusions are as follows:

(1) Through deduction, it is known theoretically that to the variational problem with weak constraints in cost functional J, its Euler equation can be discreted into difference format, by using matrix theory and difference method of partial differential equation we know that there exists optimal selection of weighting factors in the cost functional under the condition of minimal variance between analysis field and ideal field.

(2) Weighting factors selected with this method are in good coordination with numerical model and observational data and good coordination can also be achieved among weighting factors themselves. Obtaining weighting factors objectively with mathematics and calculation

mathematics methods, it excludes the arbitrariness of selection with empirical method.

(3) Experiments show that the optimal selection of FWF is good for improving the retrieval of model parameters, this method can decrease the number of experiments and take $w2$ within the good range, its values are also consistent with those by empirical method. The values taken are in the range of the latter which is much larger.

(4) Although the present paper has only discussed the optimal selection of FWF with one weak constraint with simple adjoint model, this objective method proves to be reasonable. There are still some difficulties, the stableness of dynamical calculation for example.

Nevertheless, we believe that this method can be improved through constant modification. With further study, this method can also be used in those variational problems with different constraints in different models. At present, we are making experiments on the FWF selection in simple adjoint model with radial momentum equation of cylinder coordination with more functional constraints added in, which will be discussed in detail later.

This work experiments with the NAO's and WAGU's Doppler radar data in Japan, we are grateful to the data provider. Special thanks are due to Professor Ouyang Zixiang, Professor Wu Qiguang and Professor Lin Chengsen at department of mathematics of Nanjing University for their careful checking all the formulas in this paper. The authors are also grateful to the people for their suggestions and assistance.

Appendix A

We define the first–order and second–order finite difference operators as follows:

$$\nabla_x \varphi_{i,j} = \frac{\varphi_{i+1,j} - \varphi_{i,j}}{h}, \quad \overline{\nabla}_x \varphi_{i,j} = \frac{\varphi_{i,j} - \varphi_{i-1,j}}{h},$$

$$\nabla_x \overline{\nabla}_x \varphi_{i,j} = \frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{h^2},$$

$$\nabla_x \overline{\nabla}_y \varphi_{i,j} = \frac{\varphi_{i+1,j} - \varphi_{i+1,j-1} - \varphi_{i,j} + \varphi_{i,j-1}}{h^2},$$

Where "h" represents the space step in ΔX, $\Delta Y (\Delta X = \Delta Y)$ in X and Y direction separately. To ensure the stability of difference equation, the time step Δt to h ratio must satisfy the calculation stability condition: $\left| C\frac{\Delta t}{\Delta x} \right| \leqslant 1$, where c stands for (u,v) field. Subscript "i" stands for the ith grid point in X direction and "j" for the jth grid point in Y direction.

Appendix B

1. Solution of σ_0^2

To take nine points nearby (i,j) point, the average of nine points φ_k $(k = 1,2,\cdots,9)$ is as "true" value:

$$\overline{\varphi} = \frac{1}{9} \sum_{k=1}^{9} \varphi_k$$

the difference of each point φ_k minus $\overline{\varphi}$ is as noise: $\Delta\varphi_k = \varphi_k - \overline{\varphi}$,
then

$$\sigma_0^2 = \frac{1}{9} \sum_{k=1}^{9} (\Delta\varphi_k)^2.$$

2. Solution of σ_1^2

Solution of true value is as above. The nine points true values φ_k^t ($k=1,2,\cdots,9$) are obtained separately, after that the second-order differences of φ_k^t are counted: $\nabla_x, \overline{\nabla}_x \varphi_k^t$, then

$$\sigma_1^2 = \frac{1}{9}\sum_{k=1}^{9}(\nabla_x \overline{\nabla}_x \varphi_k^t)^2.$$

3. Solution of σ_2^2

This is similar to the solution of σ_1^2, after getting φ_k^t ($k=1,2,\cdots,9$), then

$$\sigma_2^2 = \frac{1}{9}\sum_{k=1}^{9}(\varphi_k^t)^2 - \left(\frac{1}{9}\sum_{k=1}^{9}(\varphi_k^t)\right)^2.$$

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