A High–Order Compact Scheme with Square–Conservativity[1]

Ji Zhongzhen (季仲贞) and *Li Jing* (李 京)

LASG, Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing 100029

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ABSTRACT

In order to improve the accuracy of forecasts of atmospheric and oceanic phenomena which possess a wide range of space and time scales, it is crucial to design the high–order and stable schemes. On the basis of the explicit square–conservative scheme, a high–order compact explicit square–conservative scheme is proposed in this paper. This scheme not only keeps the square–conservative characteristics, but also is of high accuracy. The numerical example shows that this scheme has less computing errors and better computational stability, and it could be considered to be tested and used in many atmospheric and oceanic problems.

Key words: Square conservative scheme, Compact difference, High accuracy scheme

I. INTRODUCTION

Most computational geophysical fluid problems, such as climate modeling, numerical weather forecast and ocean current simulation, possess a wide range of time and space scales, which requires numerical schemes to have high accuracy in time and space besides long–time computational stability. However most current schemes cannot keep good stability and high computing accuracy simultaneously. Many difference schemes become less stable with the increase of computing accuracy. To solve this problem, several efficient difference schemes, such as the implicit square–conservative schemes (Zeng et al., 1982; Ji and Zeng, 1982), the explicit square–conservative schemes (Ji and Wang, 1991) and the explicit square–conservative schemes with high accuracy in time (Ji and Wang, 1994), have been proposed in the recent years. Applications show that these schemes are long–time stable. In order to improve computing accuracy in space, we analyzed many numerical schemes and found that the compact difference scheme is a high–order scheme with less stencil points. Considering the long–time computational stability of the square–conservative schemes, we combine the square–conservative scheme with the compact scheme and propose a new difference scheme which possesses both long–time computing stability and high computing accuracy. The numerical example shows that the computing results are satisfactory.

II. CONSTRUCTION OF COMPACT DIFFERENCE SCHEME

The compact difference scheme is a high accuracy scheme which uses less stencil points (Lele, 1992). For convenience of further discussion, it is necessary to introduce the main idea of the compact difference.

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Let $f(x)$ be a periodic function in $[0, L]$, $h = L/n$, $x_j = j \cdot h$ $(j = 1, 2, \cdots, n)$, $f_j = f(x_j)$ and f'_j is the approximation of $\left(\frac{df}{dx}\right)_{(x_j)}$. We have

$$\beta f'_{j-2} + \alpha f'_{j-1} + f'_j + \alpha f'_{j+1} + \beta f'_{j+2} = c\frac{f_{j+3} - f_{j-3}}{6h} + b\frac{f_{j+2} - f_{j-2}}{4h} + a\frac{f_{j+1} - f_{j-1}}{2h} \quad . \tag{1}$$

where α, β, a, b, c are derived by matching the Taylor series coefficients of various orders. These constraints are:

$$a + b + c = 1 + 2\alpha + 2\beta \qquad (2\ \text{order}) , \tag{2}$$

$$a + 2^2 b + 3^2 c = 2 \cdot \frac{3!}{2!}(\alpha + 2^2 \beta) \qquad (4\ \text{order}) , \tag{3}$$

$$a + 2^4 b + 3^4 c = 2 \cdot \frac{5!}{4!}(\alpha + 2^4 \cdot \beta) \qquad (6\ \text{order}) . \tag{4}$$

For $\alpha = \frac{3}{2}$, $b = c = 0$, $\alpha = \frac{1}{4}$, $\beta = 0$, we get the usual 4–order compact scheme:

$$f'_{j-1} + 4f'_j + f'_{j+1} = \frac{3}{h}(f_{j+1} - f_{j-1}) . \tag{5}$$

III. CONSTRUCTION OF COMPACT SQUARE–CONSERVATIVE DIFFERENCE SCHEME

The explicit square–conservative scheme was discussed in detail by Ji and Wang (1991; 1994). The construction of the high–order compact explicit square–conservative scheme is as follows. For simplicity, the scalar advection equation is used as an example.

Consider the following scalar equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 , \qquad a \leqslant x \leqslant b, \quad 0 < t \leqslant T , \tag{6}$$

$$u|_{t=0} = u_0(x), \qquad a \leqslant x \leqslant b, \tag{7}$$

Eq. (6) can be generalized as:

$$\frac{\partial u}{\partial t} + \frac{1}{3}(u^* \frac{\partial u}{\partial x} + \frac{\partial u^* u}{\partial x}) = 0 , \tag{8}$$

where u^* is the approximation of u and can be u itself.

Based on the method in Ji and Wang (1991), we have the following numerical scheme:

$$\frac{u_j^{n+1} - u_j^n}{\tau} + (A_n u)_j^n + \varepsilon_n \tau (B_n u)_j^n = 0, \ J = 1, 2, \cdots, N, \tag{9}$$

where

$$(A_n u)_j^n = \frac{1}{3}[u_j^{*n} \cdot u_j'^n + (u_u^*)_j'^n] \ . \quad (10)$$

By using the compact difference, the approximate derivatives in Eq. (10) are computed as follows:

$$Af' = \delta_0 f \ , \quad (11)$$

where f represents column vectors u_j and $(u^* u)_j$ respectively, and

$$f' = (f'_1, f'_2, \cdots, f'_N)^{\text{t}} \equiv (f'_j)_N \ ,$$

$$\delta_0 f = (f_2 - f_N, f_3 - f_1, \cdots, f_1 - f_{N-1})^{\text{t}} \equiv f_{j+1} - f_{j-1})_N \ ,$$

$$A = \frac{h}{3}\begin{bmatrix} 4 & 1 & 0 & - & - & 0 & 1 \\ 1 & 4 & 1 & - & - & 0 & 0 \\ 0 & 1 & 4 & - & - & 0 & 0 \\ & & - & - & - & & \\ 0 & 0 & 0 & - & - & 4 & 1 \\ 1 & 0 & 0 & - & - & 1 & 4 \end{bmatrix} . \quad (12)$$

In Eq. (9), B_n is the harmonious dissipative operator, ε_n is the dissipative coefficient. In Ji and Wang (1991) the following theorem has been proved.

Theorem: If A_n is an anti-symmetrical operator, B_n is a positive definite operator, $(B_n u)^n \leqslant O(1)$ and $2k_3\frac{\tau}{h} < 1$, then scheme (9) is a square-conservative difference scheme when

$$\varepsilon_n = k_1 / [(1 - \frac{\tau}{h}k_2) + \sqrt{(1 - \frac{\tau}{h}k_2)^2 - (\frac{\tau}{h}k_3)^2}\ ] \ . \quad (13)$$

where

$$\begin{cases} k_1 = \| A_n u^n \|^2 / (B_n u^n, u^n) \\ k_2 = (B_n u^n, A_n u^n) \cdot h / (B_n u^n, u^n) \\ k_3 = \| A_n u^n \| \cdot \| B_n u^n \| \cdot h / (B_n u^n, u^n) \ . \end{cases} \quad (14)$$

It has been proved (Ji and Wang, 1991) that B_n is positive definite if $B_n u^n = \dfrac{A_n u^n - A_{n-1} u^{n-1}}{\tau}$, or $B_n u^n = \dfrac{A_n \tilde{u}^{n+1} - A_n u^n}{\tau}$ and $\tilde{u}^{n+1} = u^n - \tau \cdot A_n u^n$. So the scheme (9) is explicit square-conservative as long as A_n is anti-symmetrical. For the limit of length, the proof of anti-symmetry of A_n will be gived in another paper. Now that 4-order compact difference (11) is used in the computation of 1-order derivative in Eq. (10), scheme (9) is 4-order compact explicit square-conservative.

IV. NUMERICAL EXAMPLE

In order to test the accuracy of the 4–order compact square–conservative scheme, three different numerical schemes are used to solve the following equation and their absolute errors are compared.

$$\frac{\partial u}{\partial t} + 2 \cdot \sin x \cdot \frac{\partial u}{\partial x} + u \cdot \cos x = 0 \ , \qquad x \in [0, 2\pi], \quad t > 0 \ . \tag{15}$$

$$u_{(x,0)} = \sin x \ , \qquad x \in [0, 2\pi] \ . \tag{16}$$

Under periodic boundary conditions, its analytic solution is:

$$u_{(x,t)} = \sin x \cdot \left[\frac{1 + (\mathrm{tg}\frac{x}{2})^2}{e^{2t} + e^{-2t}(\mathrm{tg}\frac{x}{2})^2} \right]^{3/2} \ . \tag{17}$$

1. *Compact Difference Scheme*

$$u_j^{n+1} = u_j^n + \Delta t(-2 \cdot \sin x_j u_j'^n - u_j^n \cdot \cos x_j) \ , \qquad j = 1, 2, \cdots, N \ , \tag{18}$$

where u'^n_j is computed by using 4–order compact difference (5).

2. *Explicit Square–conservative Scheme*

$$u_j^{n+1} = u_j^n + \Delta t[(A_n u)_j^n + \varepsilon_n \cdot \Delta t \cdot (B_n u)_j^n] \ , \ j = 1, 2, \cdots, N \ . \tag{19}$$

Considering that $2\sin x \cdot \frac{\partial u}{\partial x} + u \cdot \cos x = \sin x \frac{\partial u}{\partial x} + \frac{\partial (u \cdot \sin x)}{\partial x}$, $A_n u$ is the approximation of $-\left[\sin x \frac{\partial u}{\partial x} + \frac{\partial (u \cdot \sin x)}{\partial x}\right]$:

$$(A_n u)_j^n = -\sin x_j \frac{u_{j+1}^n - u_{j-1}^n}{2h} - \frac{u_{j+1}^n \sin x_{j+1} - u_{j-1}^n \sin x_{j-1}}{2h} \ .$$

B_n and ε_n are determined as in Ji and Wang (1991).

3. *4–order Compact Square–conservative Scheme*

$$u_j^{n+1} = u_j^n + \Delta t[(A_n u)_j^n + \varepsilon_n \cdot \Delta t \cdot (B_n u)_j^n] \ , \quad j = 1, 2, \cdots, N, \tag{20}$$

where $(A_n u)_j^n = -\sin x_j \cdot u'^n_j - (u \cdot \sin x)'^n_j$, u'^n_j and $(u \cdot \sin x)'^n_j$ are computed by using 4–order compact difference (11).

In Fig. 1, the absolute errors of three schemes at $t = 100\Delta t$ are shown. Here $h = 2\pi / 50$, $\Delta t = h / 10$. From this figure, we can see that the computing error of the compact square–coservative scheme is the smallest.

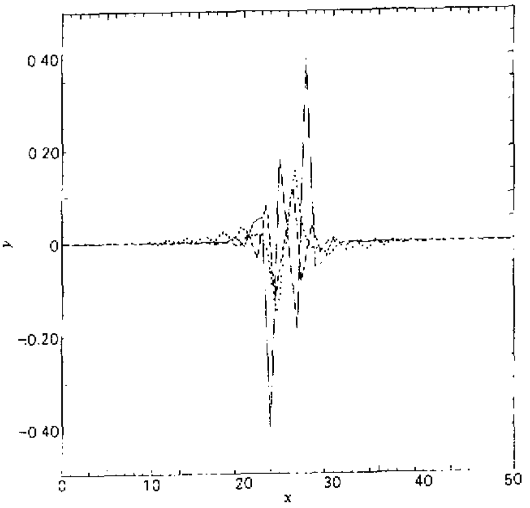

Fig. 1. Absolute errors of numerical schemes. ...... compact difference scheme —— explicit square–conservative scheme – – – 4–order compact square–conservative scheme.

The 4–order compact square–conservative scheme is also used to calculate the single soliton solutions of Kdv equation. The numerical solutions coincide with the exact solutions for a long time, which shows that this scheme is long–time computational stable.

The usefulness of the scheme proposed to atmospheric and oceanic numerical models is to be further tested.

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