

Nonlinear Stability of Zonally Symmetric Quasi-geostrophic Flow^①

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ABSTRACT

By using the conservation laws and the method of variational principle, an improved Arnol'd's second nonlinear stability theorem for the two-dimensional multilayer quasi-geostrophic model in periodic channel is obtained.

Key words: Nonlinear stability, Quasi-geostrophic flow, Improved Poincare inequality

1. Introduction

Mu et al. (1994) obtained the Arnol'd's second theorem for the multilayer quasi-geostrophic model (Pedlosky 1979). Liu and Mu (1994) extended the mentioned theorem to a more general model and simplified the nonlinear stability criterion. Here, we investigated the general model in periodic channel by different technique and obtained a nonlinear stability theorem superior to that of Mu et al. (1994). The improvement is not only in the nonlinear stability condition but also in the estimates of the disturbance bounds. The disturbance bounds obtained here is really explicitly dependent on the initial disturbance fields, while in Mu et al. (1994), the bounds involve the solution of a boundary-value problem of the system. Hence the bounds obtained in this paper are more suitable in the studying of the nonlinear saturation of instability (cf. Shepherd 1988). We also apply our theorem to the Phillips model, and the improvement is considerable. Finally, we proved Andrews' theorem for our model, that is, any basic state satisfying our nonlinear stability condition must be zonally symmetric.

2. The model of general multilayer flow

We consider the stratified fluid of $N(>1)$ superimposed layers of constant density $\rho_1 < \dots < \rho_N$, with arbitrary density jumps and mean layer depth d_i on periodic channel $D: x \in [-X, X], y \in [-Y, Y]$. The flow is governed by the multilayer quasi-geostrophic potential vorticity equations (Ripa, 1992).

$$P_{tt} + \partial(\Phi_i, P_i) = 0, \quad [i = 1, \dots, N], \quad (2.1a)$$

where $\Phi_i(x, y, t)$ is the streamfunction in layer i and

$$\bar{P} = \nabla^2 \bar{\Phi} - \mathbf{F} \mathbf{T} \bar{\Phi} + \bar{f}(y), \quad (2.1b)$$

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Where $\vec{P} = \text{col}(P_i)$, P_i is the potential vorticity in the i th layer. And $\partial(a, b) = a_x b_y - a_y b_x$ is the two-dimensional Jacobian. ∇^2 is the Laplacian on $x-y$ plane, x and y are eastward and northward coordinates, respectively, t is the time, and $\vec{f}(y) = \text{col}(f_i(y))$.

$$\begin{aligned} f_1(y) &= f + \beta y - \frac{\tau_1(y)}{d_1}, \\ f_i(y) &= f + \beta y, \quad i = 2, \dots, N-1, \quad N > 2, \\ f_N(y) &= f + \beta y + \frac{\tau_N(y)}{d_N}, \quad N \geq 2, \end{aligned}$$

where f is the representative value of the Coriolis parameter, $\tau_1(y)$ and $\tau_N(y)$ are the possible available topographies at the top and the bottom of the fluid, respectively. $\mathbf{T} = (T_{ij})$ is a matrix (cf. Liu and Mu, 1994).

$$\begin{aligned} T_{11} &= 1, \\ T_{ii} &= g_i(g_{i-1}^{-1} + g_i^{-1}), \quad [i = 2, \dots, N], \\ T_{i+1,i} &= T_{i,i+1} = -g_i g_i^{-1} < 0, \quad [i = 1, \dots, N-1], \\ T_{ij} &= 0, \quad |i - j| > 1, \\ \mathbf{F} &= \text{diag}(F_i), \quad F_i = \frac{f^2}{d_i g_i} > 0, \end{aligned}$$

where $g_i > 0$ is the buoyancy jump (not necessarily equal) across the interface between the i th and the $(i+1)$ th layer. Here we consider the rigid top and bottom boundaries, then $g_0^{-1} = 0$ and $g_N^{-1} = 0$.

The boundary conditions are

$$\Psi_x|_{y=\pm Y} = 0, \quad \int_{-X}^X \Phi_{yt}|_{y=\pm Y} dx = 0. \quad (2.1c)$$

Denote $\mathbf{K} = \text{diag}(K_i)$, $K_i = (F_i)^{-\frac{1}{2}}$, $i = 1, \dots, N$. We can see that $\sum_{i=1}^N K_i^2 \Phi_i$ can be added to an arbitrary function of time. So, to ensure the uniqueness, we assume that

$$\int_D \sum_{i=1}^N K_i^2 \Phi_i dx dy = \int_D \sum_{i=1}^N K_i^2 \Phi_{i0} dx dy, \quad (2.1d)$$

where the subscript 0 refers to the state at initial time $t = 0$.

3. Basic state and disturbance system

Suppose that $(\vec{\Phi}, \vec{P}) = (\vec{P}, \vec{Q})$ is a basic steady state of system (2.1), and there exists a constant α such that

$$\Psi_i + \alpha y = \Psi_i^\alpha(Q_i), \quad i = 1, \dots, N, \quad (3.1a)$$

where $\Psi_i^\alpha(\xi)$ is a continuous monotonic function of ξ , and there is a positive definite diagonal constant matrix

$$\mathbf{C} \equiv \text{diag}(C_1, \dots, C_N), \quad (3.1b)$$

such that

$$-\frac{d\Psi_i^\alpha}{d\xi} \geq C_i > 0, \quad [i = 1, \dots, N], \quad (3.1c)$$

holds almost everywhere.

We continue $\Psi_i^\alpha(\xi)$ to $\xi \in (-\infty, \infty)$ as follows:

$$\Psi_i^\alpha(\xi) = \Psi_i^\alpha(\min Q_i) - C_i(\xi - \min Q_i), \quad \text{for } \xi \leq \min Q_i,$$

$$\Psi_i^\alpha(\xi) = \Psi_i^\alpha(\max Q_i) - C_i(\xi - \max Q_i), \quad \text{for } \xi \geq \max Q_i,$$

so that (3.1c) holds for all $\xi \in (-\infty, \infty)$.

Define the disturbance $(\bar{\psi}, \bar{q})$ by

$$\bar{\Phi} \equiv \bar{\Psi} + \bar{\psi}, \quad \bar{P} \equiv \bar{Q} + \bar{q}, \quad (3.2)$$

and the relative disturbance $(\bar{\psi}', \bar{q}')$ by

$$\bar{\psi} \equiv \bar{\psi}_0 + \bar{\psi}', \quad \bar{q} \equiv \bar{q}_0 + \bar{q}'. \quad (3.3)$$

We have the disturbance system

$$\bar{q}' = \nabla^2 \bar{\psi}' - \mathbf{K}^{-2} \mathbf{T} \bar{\psi}', \quad (3.4a)$$

and

$$\bar{\psi}'_x|_{y=\pm Y} = 0, \quad \int_{-X}^X \bar{\psi}'_y|_{y=\pm Y} dx = 0, \quad (3.4b)$$

$$\int_D \bar{q}' dx dy = 0. \quad (3.4c)$$

By the method of decoupling system (3.1) in Liu and Mu (1992), denote $(\lambda_1, \dots, \lambda_N)$ the eigenvalues of the matrix $\mathbf{K}^{-1} \mathbf{T} \mathbf{K}^{-1}$, $(0 = \lambda_1 < \dots < \lambda_N)$, then there exists an orthogonal matrix \mathbf{L} , such that

$$\mathbf{L}^T \mathbf{K}^{-1} \mathbf{T} \mathbf{K}^{-1} \mathbf{L} = \text{diag}(\lambda_i) \equiv \Lambda, \quad (3.5)$$

where the first column of \mathbf{L} takes the form

$$\bar{\mathbf{v}} \equiv \frac{1}{\sqrt{\sum_{i=1}^N K_i^2}} \text{col}(K_i). \quad (3.6)$$

Take the transformations (superscript T denotes transpose)

$$\bar{\mathbf{p}} = \mathbf{L}^T \mathbf{K} \bar{\psi}', \quad \bar{\mathbf{b}} = \mathbf{L}^T \mathbf{K} \bar{q}'. \quad (3.7)$$

In (3.4), we obtain uncoupled disturbance system

$$\nabla^2 \bar{\mathbf{p}} - \Lambda \bar{\mathbf{p}} = \bar{\mathbf{b}}, \quad (3.8a)$$

$$\bar{\mathbf{p}}_x|_{y=\pm Y} = 0, \quad \int \bar{\mathbf{p}}_y|_{y=\pm Y} dx = 0, \quad (3.8b)$$

$$\int_D \bar{\mathbf{b}} dx dy = 0. \quad (3.8c)$$

By (3.6) and (3.7), we have

$$p_1 = \frac{1}{\sqrt{\sum_{i=1}^N K_i^2}} [\sum_{i=1}^N K_i^2 \psi'_i] ,$$

hence by (3.8abc) and (2.1d), we have

$$\int_D \bar{p} dx dy = 0 . \quad (3.8d)$$

Moreover, by (3.6)-(3.7),

$$b_1 = \frac{1}{\sqrt{\sum_{i=1}^N K_i^2}} [\sum_{i=1}^N K_i^2 q'_i] ,$$

and the conservation of the zonal impulse implies

$$\int_D y b_1 dx dy = \int_D y \nabla^2 p_1 dx dy = 0 . \quad (3.8e)$$

4. Improvement of Poincare inequality

Define the disturbance energy

$$E[\bar{p}] \equiv \int_D (|\nabla \bar{p}|^2 + \bar{p}^T \Lambda \bar{p}) dx dy , \quad (4.1a)$$

where $|\nabla \bar{p}|^2 \equiv (\bar{p}_x)^2 + (\bar{p}_y)^2$. Then the Poincare inequality derived in Mu et al. (1994) reads

$$E[\bar{p}] \leq \int_D \bar{b}^T (\lambda \mathbf{I} + \Lambda)^{-1} \bar{b} dx dy .$$

In order to improve above inequality, applying inequality (A 19) (See Appendix) to (p_1, b_1) , we have

$$\int_D |\nabla p_1|^2 dx dy \leq \frac{1}{\lambda + \mu} \int_D b_1^2 dx dy , \quad (4.1b)$$

where $\mu = \pi^2 / X^2$. Applying the usual Poincare inequality to (p_i, b_i) , $i = 2, \dots, N$, we have

$$\int_D (|\nabla p_i|^2 + \lambda_i p_i^2) dx dy \leq \frac{1}{\lambda + \lambda_i} \int_D b_i^2 dx dy , \quad [i = 2, \dots, N] . \quad (4.1c)$$

Thus we have the improved Poincare inequality by (4.1).

$$E[\bar{p}] \leq \int_D \bar{b}^T \Lambda' \bar{b} dx dy = \int_D (\mathbf{K} \bar{p}')^T \mathbf{L} \Lambda' \mathbf{L}^T (\mathbf{K} \bar{q}') dx dy = \int_D (\mathbf{K} \bar{q}')^T \mathbf{M} (\mathbf{K} \bar{q}') dx dy, \quad (4.2a)$$

where

$$\Lambda' \equiv \text{diag} \left(\frac{1}{\lambda + \mu}, \frac{1}{\lambda + \lambda_2}, \dots, \frac{1}{\lambda + \lambda_N} \right), \quad (4.2b)$$

$$\mathbf{M} \equiv \mathbf{L} \Lambda' \mathbf{L}^T. \quad (4.2c)$$

5. Conservation laws and nonlinear stability

There are well-known invariants by the conservation laws

$$(1) E[\mathbf{L}^T \mathbf{K} \bar{\Phi}] = \int_D (|\mathbf{K} \nabla \bar{\Phi}|^2 + \bar{\Phi}^T \bar{\Phi}) dx dy, \quad (\text{total energy}),$$

$$(2) \Gamma[\bar{\Phi}] \equiv 2 \sum_{j=1}^N (-1)^j \int_{-X}^X \bar{\Psi}^T \mathbf{K}^2 \bar{\Phi}_y \big|_{y=(-1)^j Y} dx, \quad (\text{circulations}),$$

$$(3) A[\bar{\Phi}] \equiv 2 \sum_{i=1}^N K_i^2 \int_D \left(\int_0^{P_i} \Psi_i^*(\xi) d\xi \right) dx dy,$$

$$(4) M[\bar{\Phi}] \equiv 2 \int_D y \sum_{i=1}^N K_i^2 P_i dx dy. \quad (\text{zonal impulse}).$$

Combining these invariants we obtain the conserved energy-Casimir functional $I[\bar{\Phi}]$,

$$I[\bar{\Phi}] \equiv E[\mathbf{L}^T \mathbf{K} \bar{\Phi}] - \alpha M[\bar{\Phi}] - \Gamma[\bar{\Phi}] + A[\bar{\Phi}]. \quad (5.1)$$

Then define the invariant \bar{I}

$$\bar{I} \equiv I[\bar{\Psi}] - I[\bar{\Phi}]. \quad (5.2)$$

Direct calculation shows that

$$\bar{I} = -E[\mathbf{L}^T \mathbf{K} \bar{\psi}] - 2 \sum_{i=1}^N K_i^2 \int_D \left(\int_{Q_i}^{P_i} (\Psi_i^*(\xi) - \Psi_i^*(Q_i)) d\xi \right) dx dy, \quad (5.3)$$

and it may also be proved that

$$E[\mathbf{L}^T \mathbf{K} \bar{\psi}_0] - \mathbf{K}[\mathbf{L}^T \mathbf{K} \bar{\psi}] = -E[\mathbf{L}^T \mathbf{K} \bar{\psi}^1] + 2 \int_D \bar{\psi}_0^T \mathbf{K}^2 \bar{q}' dx dy, \quad (5.4)$$

where

$$E[\mathbf{L}^T \mathbf{K} \bar{\psi}^1] = E[\bar{p}]. \quad (5.5)$$

Then by (3.1c) and Lebesgue integral theory, we have

$$-2 \int_{Q_i}^{P_i} (\Psi_i^*(\xi) - \Psi_i^*(Q_i)) d\xi \geq C_i q_i^2, \quad [i = 1, \dots, N]. \quad (5.6)$$

Hence, by (4.2) and (5.3)–(5.6), we have the inequality

$$\int_D (\mathbf{K}\vec{q}')^T (\mathbf{C} - \mathbf{M})(\mathbf{K}\vec{q}') dx dy + 2 \int_D (\mathbf{K}(\mathbf{C}\vec{q}_0 + \vec{\psi}_0))^T (\mathbf{K}\vec{q}') dx dy \leq \bar{I} + E[\mathbf{L}^T \mathbf{K}\vec{\psi}_0] \equiv H. \quad (5.7)$$

Now suppose that the matrix

$$\mathbf{C} - \mathbf{M} \text{ is positive definite,} \quad (5.8)$$

which ensures that its minimum eigenvalue $\nu_{\min} > 0$. From (5.7), using Holder inequality and (3.4c), we have

$$\nu_{\min} \int_D (\mathbf{K}\vec{q}')^2 dx dy - 2n \sqrt{\int_D (\mathbf{K}\vec{q}')^2 dx dy} \leq H, \quad (5.9a)$$

where

$$n \equiv \sqrt{\int_D (\mathbf{K}(\mathbf{C}\vec{q}_0^* + \vec{\psi}_0^*))^2 dx dy}, \quad (5.9b)$$

here the superscript $*$ means

$$X^* \equiv X - \frac{\int_D X dx dy}{\int_D dx dy}. \quad (5.9c)$$

Therefore, by (5.9), we obtain the bound for relative disturbance potential enstrophy

$$\sqrt{\int_D (\mathbf{K}\vec{q}')^2 dx dy} \leq \frac{n + \sqrt{n^2 + \nu_{\min} H}}{\nu_{\min}} \equiv Brp. \quad (5.10)$$

Furthermore, the bound for disturbance potential enstrophy can be obtained by

$$\begin{aligned} \int_D (\mathbf{K}\vec{q})^2 dx dy &= \int_D (\mathbf{K}\vec{q}_0)^2 dx dy + 2 \int_D (\mathbf{K}\vec{q}_0^*)^T (\mathbf{K}\vec{q}') dx dy + \int_D (\mathbf{K}\vec{q}')^2 dx dy \\ &\leq \int_D (\mathbf{K}\vec{q}_0)^2 dx dy + 2 \sqrt{\int_D (\mathbf{K}\vec{q}_0^*)^2 dx dy} Brp + (Brp)^2 \end{aligned}$$

and by (4.2a) and (5.5) we have the bound for relative disturbance energy:

$$E[\mathbf{L}^T \mathbf{K}\vec{\psi}] \leq \frac{(Brp)^2}{\min(\lambda + \mu, \lambda + \lambda_2)} \equiv Bre,$$

and the bound for disturbance energy by (5.4):

$$E[\mathbf{L}^T \mathbf{K}\vec{\psi}] \leq E[\mathbf{L}^T \mathbf{K}\vec{\psi}_0] + 2 \sqrt{\int_D (\mathbf{K}\vec{\psi}_0^*)^2 dx dy} Brp + Bre.$$

We can see that all the bounds are independent of time and tend to zero as the initial disturbances tend to zero. In this meaning we can say that the basic state is nonlinearly stable, and (5.8) is the nonlinear stability condition.

According to Liu and Mu (1994), condition (5.8) is equivalent to the matrix

$$\mathbf{M}^{-1} - \mathbf{C}^{-1} \text{ is positive definite.}$$

While

$$\mathbf{M}^{-1} = \mathbf{L}(\mathbf{A}')^{-1}\mathbf{L}^T = \mathbf{L}\mathbf{A}\mathbf{L}^T + \mu\bar{\mathbf{v}}\bar{\mathbf{v}}^T = \lambda\mathbf{I} + \mathbf{K}^{-1}\mathbf{T}\mathbf{K}^{-1} + \mu\bar{\mathbf{v}}\bar{\mathbf{v}}^T,$$

where \mathbf{I} is the identity matrix and $\bar{\mathbf{v}}$ is defined by (3.6).

Therefore the nonlinear stability condition (5.8) is simplified to the matrix

$$\lambda\mathbf{I} + \mathbf{K}^{-1}\mathbf{T}\mathbf{K}^{-1} - \mathbf{C}^{-1} + \mu\bar{\mathbf{v}}\bar{\mathbf{v}}^T \quad \text{is positive definite.} \quad (5.11)$$

Thus, we have proved

Theorem 1 Suppose that $(\bar{\Psi}, \bar{Q})$ is the basic steady state of system (2.1) with property (3.1). If (5.8) or (5.11) holds, then the basic state is nonlinearly stable to any finite-amplitude disturbances.

The nonlinear stability condition in Mu et al. (1994) is equivalent to (5.11) with its last term $\mu\bar{\mathbf{v}}\bar{\mathbf{v}}^T$ dropped, hence the nonlinear stability condition (5.11) is superior, since it is added to a positive semi-definite matrix $\mu\bar{\mathbf{v}}\bar{\mathbf{v}}^T$.

6. Two-layer model

We investigate the two-layer model in detail. The system is

$$\begin{aligned} P_{it} + \partial(\Phi_i, P_i) &= 0, \quad [i = 1, 2], \\ P_1 &\equiv \nabla^2 \Phi - F_1(\Phi_1 - \Phi_2) + f_1(y), \\ P_2 &\equiv \nabla^2 \Phi - F_2(-\Phi_1 + \Phi_2) + f_2(y). \end{aligned} \quad (6.1)$$

Denote $l \equiv 1 - \frac{\mu}{F_1 + F_2}$, then the criterion matrix of (5.11) is

$$\begin{bmatrix} \lambda + \mu + lF_1 - \frac{1}{c_1} & -l\sqrt{F_1 F_2} \\ -l\sqrt{F_1 F_2} & \lambda + \mu + lF_2 - \frac{1}{c_2} \end{bmatrix}. \quad (6.2)$$

The minimum eigenvalue of matrix (6.2) is

$$\begin{aligned} \lambda_{\min} &= \lambda + \frac{1}{2}(F_1 + F_2 + \mu) - \frac{1}{2}\left(\frac{1}{c_1} + \frac{1}{c_2}\right) \\ &\quad - \frac{1}{2}\sqrt{\left(l(F_1 - F_2) + \frac{1}{c_2} - \frac{1}{c_1}\right)^2 + 4F_1 F_2 l^2} \end{aligned} \quad (6.3)$$

and the matrix of (5.8) is

$$\begin{bmatrix} C_1 - \frac{\lambda + \mu + lF_2}{(\lambda + \mu)(\lambda + F_1 + F_2)} & -\frac{l\sqrt{F_1 F_2}}{(\lambda + \mu)(\lambda + F_1 + F_2)} \\ -\frac{l\sqrt{F_1 F_2}}{(\lambda + \mu)(\lambda + F_1 + F_2)} & C_2 - \frac{\lambda + \mu + lF_1}{(\lambda + \mu)(\lambda + F_1 + F_2)} \end{bmatrix}. \quad (6.4a)$$

Since C_1 and C_2 contain a parameter α , the determinant of (6.4) may be quadratic in α and takes its maximum for some α in some cases. Now we discuss it for later use.

Suppose that

$$C_1 = \frac{w}{G_1}, \quad (6.4b)$$

$$C_2 = \frac{1-w}{G_2}, \quad (6.4c)$$

where $G_1, G_2 > 0$ and $w \in (0,1)$.

The determinant Δ_2 of (6.4) is quadratic in w , and takes its maximum at

$$w = \frac{(\lambda + \mu)(\lambda + F_1 + F_2 + G_1 - G_2) + l(F_2 G_1 - F_1 G_2)}{2(\lambda + \mu)(\lambda + F_1 + F_2)}. \quad (6.4d)$$

The matrix (6.4a) can be rewritten as

$$\frac{1}{(\lambda + \mu)(\lambda + F_1 + F_2)} \begin{bmatrix} \frac{Z}{2G_1} & -l\sqrt{F_1 F_2} \\ -l\sqrt{F_1 F_2} & \frac{Z}{2G_2} \end{bmatrix}, \quad (6.4e)$$

where

$$Z = \lambda^2 + \lambda(\mu + F_1 + F_2 - G_1 - G_2) - l(F_2 G_1 + F_1 G_2), \quad (6.4f)$$

and the sign of Δ_2 is the same as that of

$$Z^2 - 4(F_2 G_1 F_1 G_2)l^2. \quad (6.4g)$$

6.1 Application to the Phillips model

As an application of the result of Section 6, we discuss the Phillips model

$$\Psi_i(y) = -U_i y, \quad Q_i(y) = [\beta + (-1)^{i+1} F_i U_s]y + f, \quad (6.5)$$

where $U_s = U_1 - U_2$, U_1 and U_2 are constants.

There are only four cases in all to be studied (cf. Mu et al., 1994).

Case 1. $-\frac{\beta}{F_1} < U_s < \frac{\beta}{F_2}$.

It is proved to be nonlinearly stable (cf. Shepherd, 1988, or Mu et al., 1994).

Case 2. $U_s = -\frac{\beta}{F_1}$.

Let $\alpha = U_1$, then $C_2 = \frac{1}{F_1 + F_2}$ and take C_1 sufficiently large. Therefore, by (6.3), we obtain the nonlinear stability condition

$$\lambda + \frac{1}{2}\mu > \sqrt{F_1 F_2 + (F_1 - \frac{1}{2}\mu)^2}, \quad (6.6a)$$

or equivalently

$$\lambda^2 + \mu(\lambda + F_1) > F_1(F_1 + F_2). \quad (6.6b)$$

Case 3. $U_s = \frac{\beta}{F_2}$.

Let $\alpha = U_2$, then $C_1 = \frac{1}{F_1 + F_2}$ and take C_2 sufficiently large. Therefore, by (6.3), we

obtain the nonlinear stability condition

$$\lambda + \frac{1}{2}\mu > \sqrt{F_1 F_2 + \left(F_2 - \frac{1}{2}\mu\right)^2}, \quad (6.7a)$$

or equivalently

$$\lambda^2 + \mu(\lambda + F_2) > F_2(F_1 + F_2). \quad (6.7b)$$

Case 4. $U_s < -\frac{\beta}{F_1}$, or $U_s > \frac{\beta}{F_2}$.

Now

$$C_1 = \frac{U_1 - \alpha}{F_1 U_s + \beta} \equiv \frac{w}{G_1},$$

$$C_2 = \frac{\alpha - U_2}{F_2 U_s - \beta} \equiv \frac{1-w}{G_2},$$

where $w \equiv (U_1 - \alpha) / U_s > 0$, and $G_1 = F_1 + \beta / U_s > 0$, $G_2 = F_2 - \beta / U_s > 0$. Thus we can use the results of (6.4) to obtain nonlinear stability condition in this case

$$\lambda(\lambda + \mu) > I \left[\sqrt{F_2 \left(F_1 + \frac{\beta}{U_s}\right)} + \sqrt{F_1 \left(F_2 - \frac{\beta}{U_s}\right)} \right]^2. \quad (6.8)$$

Take $\mu = 0$, then nonlinear stability conditions (6.6)–(6.8) can be turned into the results of Mu et al. (1994), which equivalent to

$$\lambda^2 > \left[\sqrt{F_2 \left(F_1 + \frac{\beta}{U_s}\right)} + \sqrt{F_1 \left(F_2 - \frac{\beta}{U_s}\right)} \right]^2. \quad (6.9a)$$

or equivalently

$$\lambda > \sqrt{F_2 \left(F_1 + \frac{\beta}{U_s}\right)} + \sqrt{F_1 \left(F_2 - \frac{\beta}{U_s}\right)}. \quad (6.9b)$$

Hence, our result is better than that of Mu et al (1994).

7. Andrews' theorem

Mu, Zeng, Shepherd and Liu (1994) have proved that any basic state satisfying their nonlinear stability theorem for zonally symmetric problem of the two-dimensional multilayer quasi-geostrophic model must itself be zonally symmetric. That is to say, Andrew's (1984) theorem holds for the basic state satisfying the nonlinear stability theorem in zonally symmetric problem. Here we use the similar technique to show that this is also true for our improved nonlinear stability theorem in this paper.

Suppose that $(\bar{\Psi}, \bar{Q})$ is the nonlinear basic state satisfying our nonlinear stability condition. Then $(\bar{\Phi} = \bar{\Psi}(x + a, y), \bar{P} = \bar{Q}(x + a, y))$ is also a steady state. The invariants (1)–(4) in Section 5 for $\bar{\Psi}$ are equal to the corresponding invariants for $\bar{\Phi}$, hence, the invariant \bar{I} of (5.2) is equal to zero. On the other hand, after transformation (3.7) on $\bar{\psi} \equiv \bar{\Phi} - \bar{\Psi}$ and $\bar{q} \equiv \bar{P} - \bar{Q}$, $(\bar{p} = \mathbf{L}^T \mathbf{K} \bar{\psi}, \bar{b} = \mathbf{L}^T \mathbf{K} \bar{q})$ satisfies (3.8), so the general Poincaré inequality holds for $\bar{\psi}$. Therefore, under the stability condition, the \bar{I} is positive definite in $\bar{q} = \bar{P} - \bar{Q}$, thus, $\bar{q} = \bar{\psi} \equiv 0$. Since a is any constant, we can see that $\bar{\Psi}_x \equiv 0$, which proves our assertion.

APPENDIX

Proof of the improved poincare inequality

Suppose that p and b satisfy the system

$$\nabla^2 p = b, \quad (\text{A1a})$$

$$p_x|_{y=\pm Y} = 0, \quad \int p_y|_{y=\pm Y} dx = 0, \quad (\text{A1b})$$

$$\int_D b dx dy = 0, \quad (\text{A1c})$$

$$\int_D p dx dy = 0, \quad (\text{A1d})$$

$$\int_D y b dx dy = \int_D y \nabla^2 p dx dy = 0, \quad (\text{A1e})$$

Decompose p into Fourier series of x

$$p = V(y, t) + W(x, y, t), \quad (\text{A2a})$$

$$W(x, y, t) \equiv \sum_{m=1}^{\infty} [U_m(y, t) \cos\left(\frac{m\pi x}{X}\right) + V_m(y, t) \sin\left(\frac{m\pi x}{X}\right)]. \quad (\text{A2b})$$

Then by (A1a)–(A1d), we obtain the conditions for $V = V(y, t)$, $U_m(y, t)$ and $V_m(y, t)$

$$\int_{-Y}^Y V dy = 0, \quad (\text{A2c})$$

$$V_y(\pm Y) = 0, \quad (\text{A2d})$$

$$U_m(\pm Y, t) = V_m(\pm Y, t) = 0, \quad [m = 1, 2, \dots].$$

Now, from (A1e), we have

$$\int_{-Y}^Y y V_{yy} dy = 0,$$

or equivalently, integrating by parts and using the boundary condition (A2d), we have

$$V(Y) - V(-Y) = 0. \quad (\text{A2e})$$

By the known Poincare inequality, it is readily to see

$$\int_D W_x^2 dx dy \geq \mu \int_D W^2 dx dy, \quad \mu = \frac{\pi^2}{X^2}, \quad (\text{A3a})$$

$$\int_D W_y^2 \geq \lambda \int_D W^2 dx dy, \quad \lambda = \frac{\pi^2}{4Y^2}, \quad (\text{A3b})$$

The key inequality to be proved is

$$\int_{-Y}^Y V_y^2 dy \geq v \int_{-Y}^Y V^2 dy, \quad v \equiv \frac{\pi^2}{Y^2}. \quad (\text{A4})$$

If (A4) is true, then by (A3) and (A4), we have

$$\begin{aligned} \int_D |\nabla p|^2 dx dy &= \int_D V_y^2 dx dy + \int_D W_x^2 dx dy + \int_D W_y^2 dx dy \\ &\geq \min(v, \lambda + \mu) \left[\int_D V^2 dx dy + \int_D W^2 dx dy \right] \end{aligned} \quad (\text{A5})$$

$$= \min(\nu, \lambda + \mu) \int_D p^2 dx dy .$$

Since the domain is the periodic channel, and $Y < \frac{\sqrt{3}}{2} X$, we have

$$\min(\nu, \lambda + \mu) = \lambda + \mu .$$

Thus, (A5) turns into

$$\int_D |\nabla p|^2 dx dy \geq (\lambda + \mu) \int_D p^2 dx dy . \quad (A6)$$

Now we shall prove (A4) by considering the variational problem

$$\nu = \min \frac{\int_{-Y}^Y V_y^2 dy}{\int_{-Y}^Y V^2 dy} , \quad \int_{-Y}^Y V^2 dy \neq 0 , \quad (A7)$$

with conditions (A2c), (A2e) and $V_y(-Y) = V_y(Y)$.

According to the variational principle, ν is the eigenvalue of the following ordinary differential equation:

$$\frac{d^2 V}{dy^2} + \nu V = 0 , \quad (A8)$$

with the boundary condition (A2e) and

$$V_y(Y) = V_y(-Y) . \quad (A9)$$

The above problem is well-posed. Its eigenvalues are discrete, and $\nu \geq 0$. The general solution of problem (A8) can be formulated as

$$V = C_1 \cos(\nu^{1/2}(y + Y)) + C_2 \sin(\nu^{1/2}(y + Y)) .$$

By the boundary condition (A9) we have

$$\begin{aligned} C_1 (\cos(2\nu^{1/2} Y) - 1) + C_2 \sin(2\nu^{1/2} Y) &= 0 , \\ -C_1 \sin(2\nu^{1/2} Y) + C_2 (\cos(2\nu^{1/2} Y) - 1) &= 0 . \end{aligned}$$

It follows from $|C_1| + |C_2| \neq 0$ that

$$[\cos(2\nu^{1/2} Y) - 1]^2 + \sin^2(2\nu^{1/2} Y) = 0 .$$

Hence

$$\cos(2\nu^{1/2} Y) = 1 ,$$

and therefore

$$2\nu^{1/2} Y = 2n\pi, \quad n = 0, 1, 2, \dots$$

From the condition $\int \nu dy = 0$, we have

$$\nu = (\pi / Y)^2 .$$

The eigenfunction $\cos \frac{\pi y}{Y}$ corresponding to the eigenvalue $v = (\pi / Y)^2$ satisfies the condition (A2d), too. Hence v is also the eigenvalue of problem (A8) under the boundary conditions (A2c), (A2e) and (A2d).

Now we can prove our improved Poincare inequality as follows:

Multiplying both sides of (A1a) by $-p$, then integrating it by parts and using (A1b), we obtain

$$\int_D |\nabla p|^2 dx dy = - \int_D p b dx dy. \quad (\text{A10})$$

Then applying the inequality (A6) and Holder inequality to (A10), we have

$$(\lambda + \mu) \int_D p^2 dx dy \leq \int_D |\nabla p|^2 dx dy \leq \sqrt{\int_D p^2 dx dy} \sqrt{\int_D b^2 dx dy}. \quad (\text{A11})$$

Hence, we have

$$\sqrt{\int_D p^2 dx dy} \leq \frac{\sqrt{\int_D b^2 dx dy}}{\lambda + \mu}. \quad (\text{A12})$$

Thus, from (A11) and (A12), we obtain the important improved Poincare inequality

$$\int_D |\nabla p|^2 dx dy \leq \frac{\int_D b^2 dx dy}{\lambda + \mu}. \quad (\text{A13})$$

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