Simplification of Potential Vorcticity and Mesoscale Quasi–balanced Dynamics Model

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ABSTRACT

The physical characteristics of mesoscale are analyzed, and results show that the unbalanced forced motion is the fundamental cause, which leads to the evolution of some important mesoscale weather systems. In this paper, an alternative asymptotic expansion method, which is quite different from the conventional Rossby–number expansion, is used to simplify the potential vorticity equation. And the quasi–balanced (QB) model based on nonlinear balance equation is derived. The QB model, which is in analogy with the quasi–geostrophic model, can describe the fundamental characteristics of the mesoscale accurately and may be used as the basis of theoretical studies on the mesoscale atmospheric dynamics.

Key words: Balanced motions, Potential vorticity, Shallow–water equations

1. Introduction

It is now widely recognized that the potential vorticity has become a very illuminating and powerful theoretical tool for studying geophysical flows on a range of scales. A fundamental problem in meteorology and oceanography is how to simplify Ertel's potential vorticity conservation on fluid particles so that it is conceptually and computationally easier to be dealt with than its more general form in the primitive equations. For the midlatitude large–scale motions, characterized by a small Rossby number in the atmosphere and ocean, quasi–geostrophy is a successful standard method for achieving this simplification. Specifically one expands all dependent variables in a series in Rossby number and solves leading order to obtain simplified, quasi–geostrophic dynamics (Kuo, 1973; Pedlosky, 1987). The quasi–geostrophic theory has become the conceptual cornerstone for the large–scale atmospheric dynamics and provided a dynamical framework for much of our understanding the slowly evolving, meteorologically significant large–scale phenomena in middle latitudes. To relax some of the restrictive assumptions in quasi–geostrophic model while still retaining simplicity relative to the primitive equations, many intermediate approximations have also been proposed (Xu, 1994; McWilliams and Gent, 1980; Hoskins, 1975). It is well known that the fundamental assumption of the quasi–geostrophic model, or any model based on geostrophy, is that the actual wind is nearly geostrophic. However, geostrophic balance has obvious limitations on timescales of the order of, or less than, the inverse of the Coriolis parameter. It is thus inaccurate for many quantitative applications and not generally applicable to the mesoscale dynamics, which may be defined as atmospheric circulations for which the Rossby

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number is of order unity. Due to the evolving processes of the mesoscale atmospheric motions are very complicated and diversified, there is still not a rational and unified theory for the mesoscale motions just like the quasi-geostrophic theory for the large-scale motions. We always hope to use the basic principles of the atmospheric dynamics to analyze the developing evolution of the mesoscale quasi-balanced motions, and further establish a theoretical framework of the quasi-balanced dynamics for the mesoscale atmospheric motions. It is well known that the shallow-water equations are frequently used in the simplified dynamical studies of the atmospheric and oceanographic phenomena. Although the mesoscale atmospheric motions are basically three-dimensional, for mathematical simplicity and dynamics clearness, and under the condition of not affecting the essentials of problems, we may still adopt the barotropic model, which could be governed by the shallow-water equations. Moreover, it still embodies the general characteristics of flows under the actions of gravity and Coriolis forces. Furthermore, when the dynamic equations are to be further applied to our understanding of the atmospheric phenomena, it is permissible to simplify them beyond the point where they can yield the acceptable weather predictions (Lorenz, 1960). The dynamical definitions of mesoscale are reviewed in Section 2. The physical analysis emphasizes the characteristics of balanced and unbalanced motions and its application to the diagnostic studies of the mesoscale weather systems in Section 3. We present an alternative asymptotic expansion method to simplify the potential vorticity directly and obtain the simplified governing equation for QB model in Section 4, and the results are summarized in Section 5.

2. Dynamical definitions of mesoscale

It is well known that the atmospheric motions systems have the multi-time and multi-space scale characteristics. Therefore, many researchers classified the division of scales for the motions of different atmospheric phenomena (Orlanski, 1975; Emanuel, 1983). However, as for the definitions of mesoscale, there exists a great difference. For example, Orlanski classified mesoscale as meso-α (200–2000 km), meso-β (20–200 km) and meso-γ (2–20 km). Emanuel defined the Lagrangian Rossby number $R_0 \equiv 2\pi / Tf_0$ (where Lagrangian time scale T is that of acceleration following a fluid parcel; f_0 is the typical value of the Coriolis parameter) being $1 \sim 10$ as mesoscale, his dynamical definition of mesoscale includes motions which the ageostrophic advection and the Coriolis acceleration are essential elements. Pielke (1984) defined the mesoscale as: (I) The horizontal scale is sufficiently large so that the hydrostatic equation can be applied; (II) The horizontal scale is sufficiently small so that the Coriolis term is small relative to the advective and the pressure-gradient forces (although it may be important), resulting in a flow field that is substantially different from the gradient wind relation above the planetary boundary layer. This definition approximates to the Orlanski's meso-β scale and the Emanuel's definition. In conclusion, a consistent dynamical definition of mesoscale might then include motions, which the Coriolis force, the pressure-gradient force and the ageostrophic advective term play an important role in the weather systems, and they have hydrostatic, anelastic and substantially ageostrophic characteristics. Its substantially ageostrophic characteristics constitute a great difference from the quasi-geostrophic characteristics of the large-scale systems; and the important roles played by the Coriolis force and the hydrostatic approximation are its great difference from that of the small-scale systems.

3. Physical analysis of mesoscale and its application

We only consider the mesoscale motion for which the Rossby number is order unity, that is, the inertia, Coriolis and pressure–gradient forces are essential elements. The governing equations of the barotropic model with a rest basic state (i.e. shallow–water equations) are

$$(\partial / \partial t + u\partial / \partial x + v\partial / \partial y)u - fv = -g\partial h / \partial x,$$
$$(\partial / \partial t + u\partial / \partial x + v\partial / \partial y)v + fu = -g\partial h / \partial y,$$
$$(\partial / \partial t + u\partial / \partial x + v\partial / \partial y)h + (H + h)(\partial u / \partial x + \partial v / \partial y) = 0, \tag{1}$$

where t is time, (u,v) are velocity components in the (x, y) coordinate directions, f is the Coriolis parameter, g is the gravitational acceleration constant (often viewed as reduced by an internal stratification factor, $\Delta\rho / \rho$, in geophysical application), H and h are the mean and perturbation fluid depths respectively. The shallow–water equations, which are the typically hyperbolic systems, govern two classes of motion with different time scales: low–frequency Rossby–type motion and high–frequency inertia–gravity motion, the former is the large–scale motion, and the latter corresponds to mesoscale. The momentum equations of Eqs. (1) can be transformed into the Lamb's formation

$$\partial u / \partial t - (f + \zeta)v = -\partial B / \partial x,$$
$$\partial v / \partial t + (f + \zeta)u = -\partial B / \partial y, \tag{2}$$

where $\zeta = \partial v / \partial x - \partial u / \partial y$ and $B = \varphi(gh) + (u^2 + v^2) / 2$ are the relative vorticity and the Bernoulli function, respectively. We may also deduce the following potential vorticity equation from Eqs. (1)

$$(\partial / \partial t + u\partial / \partial x + v\partial / \partial y)q = 0, \tag{3}$$

where

$$q \equiv \frac{f + \zeta}{H + h} \tag{4}$$

is the potential vorticity. Eqs.(2) actually describe the quasi–balanced dynamics characteristics of the mesoscale motions under the co–actions of the inertia, Coriolis and pressure–gradient forces. For the steady motions (i.e. quasi–balanced state of forces), we have

$$(f + \zeta)v = \partial B / \partial x, \quad (f + \zeta)u = -\partial B / \partial y, \tag{5}$$

the above notation is similar to the large–scale geostrophic balance and called the generalized gradient–wind balance (Noting that the gradient–wind balance referred in textbooks or literatures is a horizontal force balance between pressure–gradient, Coriolis and centrifugal forces along a fluid trajectory, which takes into account the curvature effect and neglects the particle's tangential acceleration). For the mesoscale atmospheric motion, if the quasi–balanced condition (5) is introduced (just as the large–scale quasi–geostrophic model), then the problems of the mesoscale dynamics may be simplified (Yeh and Li, 1980). In fact, if we define

$$u^* = (1 + \zeta / f)u = (1 + Ro)u, \quad v^* = (1 + \zeta / f)v = (1 + Ro)v, \tag{6}$$

we name u^* and v^* as the equivalent geostrophic winds, then (5) can be written as

$$fu^* = -\partial B / \partial y, \quad fv^* = \partial B / \partial x. \tag{7}$$

This notation is the same as the geostrophic balance in the mathematical formula. f may be considered as a constant for the mesoscale, from (7) we may get the following relations,

$$D^* = 0, \quad \zeta^* = \nabla_h^2 B, \tag{8}$$

where D^* and ζ^* are the divergence and the vorticity of the equivalent geostrophic winds, respectively, and ∇_h^2 is the Laplacian operator. It can be seen clearly that although the quasi-balance ($\zeta^* \approx \nabla_h^2 B$) is identical to quasi-geostrophy ($\zeta \approx \nabla_h^2 \varphi$) in the mathematical formula, it denotes a quasi-balanced state of forces among the inertia, Coriolis and pressure-gradient forces. And, under the condition of steady motions ($\partial / \partial t = 0$), we also have the following relations (obtained from (3) and (5))

$$u\partial q / \partial x + v\partial q / \partial y = 0, \quad u\partial B / \partial x + v\partial B / \partial y = 0, \tag{9}$$

those show that under the condition of stationary motion, the isoline of potential vorticity and isoenergy are streamlines. Zeng (1979) also pointed out that only all flow parcels in moving processes did not advect energy and potential vorticity, the motion is steady. If anyone's conditions did not be met, then the advections will absolutely lead to the atmospheric motions change. However, the studies of stationary flow field (balanced motions) make sense. Actually, the factor, which causes the flow field changes with time; namely, destroys the condition of balanced motions. Therefore, analyzing the condition of steady flow field, from another point of view, it depicts the factor's role in governing evolution processes. Furthermore, the physical analyses of the mesoscale quasi-balanced motions also have practical application, if we find out the balanced and unbalanced relations in the atmosphere, then the results can be applied to the diagnostic analyses of the occurring and developing dynamics mechanism of a weather system. Both observed facts and theoretical researches show that the formation and the development of the rainstorm systems have very close relations with the distribution and the evolution of the environmental vorticity and divergence. Accordingly, it is much more convenient to use the vorticity and divergence equations to diagnose and analyze the structures of flow field than other equations (e.g. the momentum equations). Sun (1982) used the divergence equation to study the rainstorms in southern China. Based on the facts that the occurrence of heavy rainfall is often related to the low-level jets, his studies mainly focused the influence of the non-uniform distribution of horizontal wind field on the divergence changes, and he proposed that

$$A = \partial / \partial x(u\partial u / \partial x + v\partial u / \partial y) + \partial / \partial y(u\partial v / \partial x + v\partial v / \partial y) \tag{10}$$

is used to analyze the horizontal divergence changes and its relations with the upcoming rainstorms, and the practical operations also obtain relatively good results. Meanwhile, he also pointed out that there often exist the ageostrophic parts in the meso- and small-scale atmospheric motions. And from his computing results we see that the value of A seems to be small ($1 \sim 2 \times 10^{-9} s^{-2}$), however, the ageostrophic vorticity $\zeta' = (\zeta - \nabla_h^2 \varphi)$ in the real atmosphere is generally large (Wang and Sun, 1988). When ζ' exceeds A, even though A is positive, it cannot make the low-level convergence increase. In this way, the magnitude of A cannot reflect the divergence changes accurately. Therefore, using A alone to diagnose the divergence changes sometimes will cause numerous errors. On the other hand, the divergence equation can be obtained from Eqs. (1)

$$\partial D / \partial t = (\zeta - \nabla_h^2 \varphi) - A - \beta u, \tag{11}$$

where D is the horizontal divergence, $\beta = \partial f / \partial y$ is the Rossby parameter. If we consider the mesoscale motions, then the βu term of (11) may be neglected and the divergence equation (11) may be simplified to

$$\partial D / \partial t = (f\zeta - \nabla_h^2 \varphi) - A = f\zeta' - A. \tag{12}$$

Apparently, the condition, which A can describe the divergence changes accurately, is that the ageostrophic vorticity ($f\zeta'$) is small. However, in the activity areas of mesoscale, the divergence changes caused by the ageostrophic vorticity are large terms in the equation (12). And from the computing results, we can see that the divergence variations caused by the nonlinear terms also cannot be neglected, which is usually equivalent to the ageostrophic vorticity's role (Wang and Sun, 1988). It is obvious that only using A to determine the divergence changes is restricted. If we introduce the equivalent geostrophic wind $\vec{V}_h^*$ (u^*, v^*), accordingly, the divergence equation (12) may be rewritten as

$$\partial D / \partial t = f\zeta^* - \nabla_h^2 B. \tag{13}$$

It can be seen clearly from (13) that the physical content is described by the divergence equation. For the mesoscale motion, the cause of drastic changes in the divergence field is that the equivalent vorticity ζ^* and energy B fields do not satisfy the quasi-balanced relation $f\zeta^* \approx \nabla_h^2 B$. This is quite similar to the situation which the vorticity and geopotential fields of the large-scale motion not satisfying the quasi-geostrophy will lead to the drastic divergence changes. Because the two terms in the right of (13) are easily calculated, and its credibility is good, (13) can be used in the practical operations to analyze the evolving trend of the divergence field. Comparison of (12) and (13) indicates that for the mesoscale atmospheric motion, under the condition of the quasi-geostrophic approximation ($f\zeta \approx \nabla_h^2 \varphi$), maybe there are still relatively strong changes in the divergence field (A's role). However, under the condition of the quasi-balanced approximation ($f\zeta^* \approx \nabla_h^2 B$), the divergence field has not clear variations ($\partial D / \partial t \approx 0$). Thus it can be seen that $f\zeta^* \approx \nabla_h^2 B$ depicts the quasi-balanced characteristics of the mesoscale atmospheric motion much more accurately than the quasi-geostrophy ($f\zeta \approx \nabla_h^2 \varphi$), it integrates A and ageostrophic ($f\zeta'$) terms. Thus the application of (13) to diagnose the divergence changes maybe gets a better result. If ($f\zeta^* - \nabla_h^2 B$) < 0, then the unbalanced forced motions will excite convergence grow; on the contrary, if ($f\zeta^* - \nabla_h^2 B$) > 0, then the unbalanced forced motions will excite divergence increase. Numerous example analyses showed that the rainstorm systems often occur at the area where the intense convergence grows in the low-level atmosphere. Hence, under a favorable environment, if the negative value of ($f\zeta^* - \nabla_h^2 B$) in the low-level atmosphere reaches a certain intensity, then the unbalanced motions will force the low-level atmosphere to reach a certain intensity, then the unbalanced motions will force the low-level convergence to intensify swiftly and subsequently cause rainstorms. This maybe is the dynamic mechanism for the unbalanced forced motions in the atmosphere triggering the rainstorm weather, so we can use the distribution of ($f\zeta^* - \nabla_h^2 B$) values to quantitatively describe the dynamical factors, which excite the rainstorm weather systems. Chen (1993) used the divergence equation, which is in analogy with (13), to investigate the dynamical mechanism of the unbalanced motions in the atmosphere triggering the heavy rainfall weather systems. The above discussions mainly focus on the application of the divergence equation to diagnose and analyze the evolution of the mesoscale weather systems, and obtained ideal results. The reason is that the horizontal divergence is an extremely active physical factor in the development of the mesoscale weather systems and the occurrence of rainstorm than that of the vorticity. The low-level convergence ceaselessly producing and subsequently exciting the inertia-gravity waves conduce to triggering the development of the mesoscale weather systems and the formation of the torrential rains (Chao, 1980). Accordingly, in order to understand the dynamics essentialities of the heavy rainfall systems, it is absolutely necessary to investigate the divergence's evolution of wind field. In fact, through the discussions of the coupling interactions between the vertical vorticity and horizontal divergence equations, we also find out that when the divergence change becomes an important physical quantity in the process of a rainstorm, there will appear the coupling oscillations between the vertical vorticity and the horizontal divergence. The main physical mechanism of these oscillations is due to the existence of ageostrophic motions and nonlinear interactions, if only these oscillations exist, then the mutual adjustments will continue. Obviously, these oscillations possess the characteristics of the inertia-gravity waves, and certainly conduce to triggering convection and accelerating rainstorm development (He, 1989). Moreover, these kind of nonlinear interactions are very important to the developing evolution of the mesoscale weather systems. For example, some theoretical studies have shown that the formation and the evolution of the squall lines are those nonlinear evolving processes of the solitary waves, which can be described by the famous *K–dV* equation (Li, 1981). In brief, there exists a quasi-balanced relation $f\zeta^* \approx \nabla_h^2 B$ (from the deducing process, we can see that it actually describes the quasi-balanced state among the inertia, Coriolis and pressure-gradient forces) in the mesoscale motion, which describes the quasi-balanced characteristics of the mesoscale more accurately than the quasi-geostrophic balance ($f\zeta \approx \nabla_h^2 \varphi$). Therefore, it is appropriate to introduce the quasi-balanced relation $f\zeta^* \approx \nabla_h^2 B$ in the diagnostic analysis of the mesoscale atmospheric motion. And the nonbalance ($f\zeta^* \neq \nabla_h^2 B$) between its energy and vorticity field will cause the divergence change. So we can use (13) to diagnose the evolving trends of the divergence field, and further study the unbalanced motions in the atmosphere which can excite the development of a mesoscale weather system and diagnose the dynamical mechanism of the occurrence of the heavy rainfall. In order to understand the dynamics of the mesoscale weather systems (e.g. the heavy rainfall processes), it is essential to investigate the internal balanced and unbalanced motions in the atmosphere, which drive the evolution of the divergence field.

4. Potential vorticity simplification and quasi-balanced model

From the point of view of dynamics, the different space-scale motions in the atmosphere, they undergo different kind of forces. For the large-scale motion, its main characteristic is the quasi-balanced state of forces between the Coriolis and pressure-gradient forces which leads to the quasi-geostrophy, which is also the physical basis of fundamental assumption in the quasi-geostrophic model. When this quasi-balanced state breaks down, firstly, the velocity and pressure fields undergo a drastic adjustment, that is the so-called geostrophic adaptation stage; then enter the quasi-geostrophic evolution processes and finally enter its quasi-steady stage. While the dynamical characteristics of mesoscale motion are the quasi-balanced state of forces among the inertia, Coriolis and pressure-gradient forces, there also exist the adjustment stage, the developing stage under the quasi-balanced state of forces and the quasi-steady stage (Yeh and Li, 1964). The real development of a weather system large- or small-scale, which can be observed in the atmosphere, is actually the evolutional processes undergoing in quasi-balanced state of forces. And this nonlinear evolutional process is the main driving force for the development and the evolution of the weather system. For the mesoscale atmospheric motion, the physical analyses show that the dynamic processes are basically controlled by the co-actions of the inertia, Coriolis and pressure-gradient forces. That is to say, the mesoscale motion usually proceeds under the quasi-balanced state of forces among the inertia, Coriolis and pressure-gradient forces.

It is well known that, for the large-scale atmospheric motions, the actual winds are generally decomposed into:

$$u = u_g + u_a, \quad v = v_g + v_a, \tag{14}$$

where (u_g, v_g) are the geostrophic components of the winds, (u_a, v_a) are the ageostrophic components. Moreover, $u_a << u_g$, $v_a << v_g$ and (u_g, v_g) satisfy the following relations

$$\begin{aligned} & fv_g = \partial\varphi / \partial x, \quad fu_g = -\partial\varphi / \partial y, \\ & \partial u_g / \partial x + \partial v_g / \partial y = 0 \end{aligned} \tag{15}$$

for the large-scale motions, $Ro < 1$, the relative vorticity is generally smaller than the geostrophic vorticity, that is

$$\zeta \sim U / L \sim (Ro) f \Rightarrow \zeta < f \quad \text{(large - scale)}. \tag{16}$$

Since $h << H$, and the potential vorticity (4) may be reduced to

$$\begin{aligned} \frac{f+\zeta}{H+h} &= \frac{f(1+\zeta/f)}{H(1+h/H)} = \frac{f}{H}(1+\frac{\zeta}{f})(1-\frac{h}{H}+\cdots) = \frac{f}{H}(1+\frac{\zeta}{f}-\frac{h}{H}+\cdots) \\ &\approx \frac{1}{H}(f+\zeta-f\frac{h}{H}) \approx \frac{1}{H}(f+\zeta^{(0)}-\frac{f_0^2}{gH}\frac{gh}{f_0}) = \frac{1}{H}(f+\zeta^{(0)}-\lambda_0^2\psi_g), \end{aligned} \tag{17}$$

where $\lambda_0^2 = f_0^2 / gH = f_0^2 / c_0^2$, $\psi_g = gh / f_0$ is the geostrophic streamfunction, and $\zeta^{(0)} = \nabla_h^2 \psi_g$. In the above expansion series, we only retain the two terms and neglect the small term $(\zeta / f)(h / H)$. Neglecting the constant coefficient $1 / H$, we have the quasi-geostrophic potential vorticity $q = f + \zeta^{(0)} - \lambda_0^2\psi_g = f + \nabla_h^2\psi_g - \lambda_0^2\psi_g$, for a midlatitude tangent plane with the Coriolis parameter $f = f + f_0 + \beta y$. And the quasi-geostrophic potential vorticity equation is $\partial q / \partial t + J(\psi_g, q) = 0$. Many researchers used the asymptotic theory to expand all dependent variables in a serious in Rossby number and solve leading order to obtain the simplified, quasi-geostrophic dynamics (Pedlosky, 1987). Here we use alternative method to simplify the potential vorticity and obtain similar result.

Similar to the above procedure, for the mesoscale atmospheric motions, the actual winds are decomposed into:

$$u = u_b + u_n, \quad v = v_b + v_n \tag{18}$$

where (u_b, v_b) are the balanced and non-divergent components of the winds, (u_n, v_n) the nonbalanced components (which contain the divergent part). Moreover, $u_n << u_b$, $v_n << v_b$ and (u_b, v_b) satisfy the following relations

$$(f+\zeta_b)v_b = \partial B / \partial x, \quad (f+\zeta_b)u_b = -\partial B / \partial y,$$
$$\partial u_b / \partial x + \partial v_b / \partial y = 0. \tag{19}$$

Comparison of (15) and (19) indicates that the geostrophic balance of the large-scale motion is replaced by the generalized gradient-wind balance of the mesoscale motion, but the mesoscale motion still possesses the horizontal nondivergence characteristic. From the non-divergent relations, we may introduce a streamfunction $\psi_b : u_b = -\partial\psi_b / \partial y$, $v_b = \partial\psi_b / \partial x$, the momentum equations in Eqs. (1) can be transformed into the following so-called nonlinear balance equation

$$\nabla_h^2 \varphi = f\nabla_h^2 \psi_b + 2J(\partial\psi_b / \partial x, \partial\psi_b / \partial y), \tag{20}$$

where $J(A,B) \equiv (\partial A / \partial x)(\partial B / \partial y) - (\partial A / \partial y)(\partial B / \partial x)$ is the Jocobian operator. By a formal scaling analysis of mesoscale, $f$may be considered as a constant and $Ro \approx 1$, so we have the following relation

$$\zeta \sim U / L \sim (Ro) f \Rightarrow \zeta \approx f. \quad \text{(meso-scale)} \tag{21}$$

For the mesoscale atmospheric motion, its potential vorticity (4) may be deduced to

$$\frac{f+\zeta}{H+h} = \frac{f+\zeta}{H(1+h/H)} = \frac{f+\zeta}{H}(1 - \frac{h}{H} + \cdots) = \frac{1}{H}[f+\zeta - (f+\zeta)\frac{h}{H} + \cdots]$$
$$\approx \frac{1}{H}[f+\zeta^{(0)} - \frac{f_0+\zeta^{(0)}}{gH}(gh)] = \frac{1}{H}(f+\zeta^{(0)} - \frac{f_0+\zeta^{(0)}}{c_0^2}\varphi^{(0)}), \tag{22}$$

where $\varphi^{(0)} = gh$. Neglecting the constant coefficient $1/H$ and using (20) to obtain the quasi-balanced potential vorticity

$$q_b = f+\zeta^{(0)} - \frac{f+\zeta^{(0)}}{c_0^2}\varphi^{(0)}$$
$$= f + \nabla_h^2\psi_b - \frac{f+\nabla_h^2\psi_b}{c_0^2}[f\psi_b + 2\nabla_h^{-2}J(\partial\psi_b / \partial x, \partial\psi_b / \partial y)], \tag{23}$$

so the governing equation of the QB model is the following quasi-balanced potential vorticity equation,

$$\partial q_b / \partial t + J(\psi_b, q_b) = 0, \tag{24}$$

where the streamfunction ψ_b is determined by the nonlinear balance equation (20), the potential vorticity $q_b = q_b(\psi_b)$ is a function of ψ_b. Comparison of (17) and (22) indicates that, for the mesoscale motions, the relative vorticity ζ, which is neglected in the large-scale potential vorticity compared with f_0, must be retained. The equation (24) is consistent with the results of theoretical analysis (Yeh and Li, 1982). That is, for the adiabatic and frictionless atmospheric motions, large- or small-scale, there exists a potential vorticity equation, which depicts the developing and evolving processes of the atmospheric motions under the quasi-balanced state of forces and can be reduced to $\partial q / \partial t + J(\psi, q) = 0$ formation. Vallis (1996) also obtained the potential vorticity equation (24) in his studies, but the potential vorticity obtained by Vallis is

$$q = f+\zeta^{(0)} - (f_0^2 / gH)[\psi^{(0)} + 2\nabla_h^{-2}J(\psi_x^{(0)}, \psi_y^{(0)})], \tag{25}$$

where streamfunction $\psi^{(0)}$ is determined by the nonlinear balance equation. This is in analogy to our results, but (23) is more accurate than (25), in fact, the potential vorticity obtained by Vallis is only the geostrophic potential vorticity plus the small Jacobian term. From comparison between the quasi-balanced and quasi-geostrophic models, we can clearly see that the physical basis of the fundamental assumption in the QB model is the nonlinear balance of the mesoscale motion, it replaces the geostrophic balance in the quasi-geostrophic model.

5. Conclusion

The physical analyses have shown that there exists a quasi-balanced relation in the mesoscale motions, $f\zeta^* \approx \nabla_h^2 B$. Although the quasi-balanced relation is identical to the quasi-geostrophy ($f\zeta \approx \nabla_h^2 \varphi$) in the mathematical formula, it denotes a quasi-balanced state among the inertia, Coriolis and pressure-gradient forces and describes the quasi-balanced characteristics of the mesoscale motion more accurately than the quasi-geostrophic balance. And the nonbalance between the energy and vorticity field ($f\zeta^* \neq \nabla_h^2 B$) will cause the divergence change. So we can use the divergence equation (13) to study the unbalanced forced motions in the atmosphere which can excite the development of a mesoscale weather system, and diagnoses the dynamical mechanism of the occurrence of the heavy rainfall. In order to understand the dynamics of the mesoscale weather system (e.g. the heavy rainfall processes), it is essential to investigate the internal balanced and unbalanced motions in the atmosphere, which drive the evolution of the divergence field.

The mesoscale dynamic equations are the basis of theoretical studies on the mesoscale atmospheric dynamics. Therefore, according to the characteristics of the mesoscale motion, the formal scaling analysis and the asymptotic theory are applied to simplify the governing atmospheric dynamics equations, which are based on fluid dynamics and thermodynamics. So they can describe the fundamental characteristics of the mesoscale motion more accurately, these procedures are essential for the development of the mesoscale dynamics. Based on observational data and theoretical analyses, we know that the atmospheric motions, large- or small-scale, usually proceed under the quasi-balanced state of forces. When this quasi-balanced breaks down, there must be a mechanism to bring the motions back to the state of quasi-balanced forces swiftly. The real development of a weather system, which can be observed in the atmosphere, is actually the evolutional processes undergoing in the quasi-balanced state of forces. And this nonlinear evolutional process is the main driving force for the development and the evolution of the weather system. So this quasi-balanced characteristics may be introduced into the momentum equations to simplify mathematical processing of dynamics problems. In fact, the quasi-geostrophic model is often used to investigate the large-scale atmospheric dynamics and a great success is achieved. It is well known that, the fundamental assumption of the quasi-geostrophic model, or any model based on geostrophy, is that the actual wind is nearly geostrophic. The corresponding assumption made in deriving the QB model is that the wind is nearly generalized gradient-wind balance and non-divergent. In this paper, the QB model, which is in analogy with the quasi-geostrophic model, is derived. Many comparative analyses and numerical simulations have shown that the nonlinear balance equation has high-degree validity (Gent and McWilliams, 1982; Raymond, 1992; Whitaker, 1993; Gent, et al. 1994). So the

QB model, which is based on the nonlinear balance, can describe the basic characteristics of the mesoscale motion accurately and may be used as the basis of the theoretical studies on the mesoscale atmospheric dynamics. Moreover, the numerical experiments on the potential vorticity evolution and inversion for the QB model, and the application of the QB model proposed in this paper to numerical simulations of some mesoscale systems will be published elsewhere.

REFERENCES

Chao, J.P., 1980: The gravitational wave in non-uniform stratification atmosphere and its preliminary application for the prediction of heavy rainfall. *Scientia Atmospherica Sinica*, **4**, 230–235 (in Chinese).

Chen, Z.M., 1993: Study of some problems in the application and simplification of divergence equation, *Acta Atmospherica Sinica*, **17**, 540–547 (in Chinese).

Emanuel, K.A., 1983: On the dynamical definition(s) of mesoscale. Mesoscale Meteorology: Theories, Observations and Models, Reidel Publishing Company, 1– 11.

Gent, P.R. and J.C. McWilliams, 1982: Intermediate model solutions to the Lorenz equations: Strange attractors and other phenomena. *J. Atmos. Sci.*, **39**, 3–13.

Gent, P.R., J.C. McWilliams and C. Snyder, 1994: Scaling analysis of curved fronts: Validity of the balance equations and semigeostrophy. *J. Atmos. Sci.*, **51**, 160–163.

He, H.Y., 1989: Internal inertia–gravity waves in a non–uniformly stratified atmosphere and their role in initiating convection. *J. Tropical Meteor.*, **5**, 8–17 (in Chinese).

Hoskins, B.J., 1975: The geostrophic momentum approximation and the semi– geostrophic equations. *J. Atmos. Sci.*, **32**, 233–242.

Kuo, H. L., 1973: Dynamics of quasi-geostrophic flows and instability theory. *Adv. Appl. Mech.*, **13**, 248–330.

Li, M.C., 1981: Nonlinear processes of formation of the squall lines in the atmosphere and KDV equation. *Scientia Sinica*, 341–350.

Lorenz, E.N., 1960: Maximum simplification of the dynamic equations. *Tellus*, **12**, 243–254.

McWilliams, J.C. and P.R. Gent, 1980: Intermediate models of planetary circulations in the atmosphere and oceans. *J. Atmos. Sci.*, **37**, 1657–1678.

Orlanski, I., 1975: A rational subdivision of scales for atmospheric process. *Bull. Amer. Meteor. Soc.*, **56**, 527–530.

Pedlosky, J., 1987: *Geophysical Fluid Dynamics*, 2nd ed., Springer– Verlag, 200–210.

Pielke, R.A., 1984: *Mesoscale Meteorological Modeling*. Academic Press, P10.

Raymond, D.J., 1992: Nonlinear balance and potential–vorticity thinking at large Rossby number. *Q. J. R. Meteor. Soc.*, **118**, 987–1015.

Sun, S.Q. 1982: The dynamic effect of wind field in low–level on the formation of heavy rainfall. *Scientia Atmospherica Sinica*, **6**, 394–404 (in Chinese).

Vallis, G.K., 1996: Potential vorticity inversion and balanced equations of motion for rotating and stratified flow. *Q. J. R. Meteor. Soc.*, **122**, 291– 322.

Wang, Z.X. and S.Q. Sun, 1988: The relationship between environmental vorticity and divergence field associated with heavy rain systems. *Acta Meteorologica Sinica*, **46**, 492–496 (in Chinese).

Whitaker, J.S., 1993: A comparison of primitive and balance equation simulations of baroclinic waves. *J. Atmos. Sci.*, **50**, 1519–1530.

Xu, Q., 1994: Semibalance model – connection between geostrophic–type and balanced–type intermediate models. *J. Atmos. Sci.*, **51**, 953–970.

Yeh, T.C. and M.C. Li, 1964: The adaptation between the pressure and the wind field in the meso– and small–scale motion. *Acta Meteorologica Sinica*, **34**, 409–423 (in Chinese).

Yeh, T.C. and M.C. Li, 1980: On the multi–time scales of different kind motion in the atmosphere. *Proceedings of Second Conference of Numerical Forecasting*. Beijing, Science Press, 181–192.

Yeh, T.C. and M.C. Li, 1982: On the characteristics of the scales of the atmospheric motions. *J. Meteor. Soc. Japan*, **60**, 16–23.

Zeng, Q.C., 1979: *The Physical–Mathematical Basis of Numerical Weather Prediction*. Vol. I, Beijing, Science Press, 543pp (in Chinese).