Baroclinic Instability in the Generalized Phillips' Model
Part II: Three–layer Model①

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ABSTRACT

The nonlinear stability of the three–layer generalized Phillips model, for which the velocity in each layer is constant and the top and bottom surfaces are either rigid or free, is studied by employing Arnol'd's variational principle and a prior estimate method. The nonlinear stability criteria are established. For comparison, the linear instability criteria are also obtained by using normal mode method, and the influences of the free parameter, β parameter and curvature in vertical profile of the horizontal velocity on the linear instability are discussed by use of the growth rate curves.

The comparison between the nonlinear stability criterion and the linear one is made. It is shown that in some cases the two criteria are exactly the same in form, but in other cases, they are different. This phenomenon, which reveals the nonlinear property of the linear instability features, is explained by the explosive resonant interaction (ERI). When there exists the ERI, i.e., the nonlinear mechanisms play a leading role in the dynamical system, the nonlinear stability criterion is different from the linear one; on the other hand, when there does not exist the ERI, the nonlinear stability criterion is the same as the linear one in form.

Key words: The generalized Phillips' model, Linear instability, Nonlinear stability

1. Introduction

Baroclinic instability is the process by which the available potential energy of a rotating stratified fluid may be converted into the kinetic energy of a growing disturbance. Through the generation of eddies and the subsequent interaction between these eddies and the mean flow, baroclinic instability is an important feature of the dynamics of large–scale geophysical systems. The first complete theoretical explanations of the phenomenon were given independently by Charney (1947) and Eady (1949), who used simple models to explain the large–scale waves observed in the atmosphere. They obtained the linear instability criteria by using the normal mode method. The linear theory has been widely extended to include effects such as dissipation, horizontal shear, and bottom topography and so on. For the Eady model, Mu and Shepherd (1994) and Liu and Mu (1996) studied its nonlinear stability, and Li et al. (1998a,b) discussed the linear and nonlinear stability for the generalized Eady model.

Phillips (1951) utilized the two homogeneous layers instead of a continuous system to investigate the baroclinic instability. The model called Phillips model is a two–layer homogeneous fluid with the surfaces of top and bottom being rigid. This model had the advantage of simplifying the actual fluid motions while retaining the essential dynamics of the instability.

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The model gave the same stability features as the corresponding Eady–type model with qualitative agreement. Phillips (1951) studied the linear instability of this model by using the normal mode method, and a comprehensive review can be found in Pedlosky (1987). Mu et al. (1994) and Mu (1998) studied the nonlinear stability of the Phillips model by employing Arnol'd method. For the two–layer generalized Phillips model, where the surfaces can be either free or rigid, Li and Mu (1996) discussed the linear and nonlinear stability. The linear instability criterion and nonlinear stability criterion were obtained by using the normal mode method and Arnol'd's variational principle and a prior estimate method, respectively. It is shown that the nonlinear stability criterion is the same as the linear one in form.

The disadvantage of the two–layer model is that there are only two vertical degrees of freedom, therefore the model is only applicable to systems with simple and similar vertical distributions of velocity and density. The aim of the study on the three–layer generalized Phillips model presented here is to investigate effects not covered by the two–layer model. As more vertical layers are included in such a homogeneous model, the vertically integrated PV (potential vorticity) flux must still be zero, but the question how this flux is distributed in the vertical becomes an issue. One of our purposes in this study is to investigate the vertical structure of the PV flux in the simplest nontrivial context. In the three–layer model, there are two "temperature gradients", corresponding to the vertical shears across the two interfaces. The issue is of potential interest with regard to dynamics of the greenhouse warming, since general circulation models suggest that the equilibrium climate consistent with an increased concentration of the greenhouse gases has smaller lower–tropospheric temperature gradients but larger upper–tropospheric gradients than the present climate (e.g., Manabe and Wetherald, 1975).

In this paper, the linear instability and nonlinear stability for the three–layer generalized Phillips model are studied. In Section 2, the basic state and governing equation are established. Equal depth layers with equal density steps are chosen, and the surfaces can be either free or rigid. Velocity steps may differ, and this system can be used to simulate a parabolic continuous velocity profile. The linear stability of this model is investigated by employing the normal mode method, and linear stability criteria are obtained in Section 3, moreover the influences of the free parameter, β parameter and curvature in vertical profile of the horizontal velocity on instability are discussed by using the growth rate curves. In Section 4, the nonlinear stability criterion for this model is obtained by employing the results of Mu et al. (1994). The comparison between the nonlinear stability criterion and the linear one shows that they are identical in some cases and different in other cases. It is natural because the stability of a linearized system does not imply the stability of original nonlinear system, and the nonlinear mechanisms are capable of amplifying infinitesimal initial disturbances. This phenomenon is explained by using the theory of explosive resonant interaction (ERI).

2. The Model

The governing equation of the three–layer quasi–geostrophic model is as follows (Pedlosky, 1987; Ripa, 1992; Zeng, 1979):

$$\frac{\partial P_i}{\partial t} + \partial(\Phi_i, P_i) = 0 \qquad i = 1,2,3, \tag{1}$$

$$P_i = \nabla^2 \Phi_i - d_i^{-1} \sum_{j=1}^{3} T_{ij} \Phi_j + f_0 + \beta y, \qquad i = 1,2,3, \tag{2}$$

where Φ_i, P_i are the stream function and the potential vorticity in the ith layer, respectively, d_i is the height of the ith layer, f_0 (constant) is the Coriolis parameter, $\partial(f,g)= f_x g_y - f_y g_x$ is the two-dimensional Jacobian, ∇^2 is the two-dimensional Laplacian operator, and T_{ij} is the element of the following matrix,

$$T=[T_{ij}]=\begin{bmatrix} f_0^2(g_0^{-1}+g_1^{-1}) & -f_0^2 g_1^{-1} & 0 \\ -f_0^2 g_1^{-1} & f_0^2(g_1^{-1}+g_2^{-1}) & -f_0^2 g_2^{-1} \\ 0 & -f_0^2 g_2^{-1} & f_0^2(g_2^{-1}+g_3^{-1}) \end{bmatrix}, \tag{3}$$

where g_i is the buoyancy jump across the interface between the ith and $(i+1)$th layer, and if the top (or bottom) surface is rigid, then $g_0^{-1}=0(g_3^{-1}=0)$; and if the top (or bottom) surface is free, then $g_0^{-1}>0(g_3^{-1}>0)$. Without the loss of generality, we just consider the case that the bottom surface is rigid and the top surface is free, i.e., $g_3^{-1}=0$ and $g_0^{-1}>0$. Suppose that $g_1^{-1}=g_2^{-1}>0$, we define

$$\alpha=g_0^{-1}/g_1^{-1} \tag{4}$$

as the *free surface parameter*. If the top surface is rigid, then $\alpha=0$; and if the top surface is free, then $\alpha>0$.

The horizontal domain under consideration is a periodic channel; the boundary conditions are the usual ones of non-normal flow and conservation of circulation in each layer, namely,

$$\left.\frac{\partial\Phi_i}{\partial x}\right|_{y=-Y,Y}=0,\quad \frac{d}{dt}\left\{\int_{-\pi L}^{\pi L}\frac{\partial\Phi_i}{\partial y}\,dx|_{y=-Y,Y}\right\}=0,\qquad i=1,2,3. \tag{5}$$

Supposing that (Ψ_i, Q_i) is the basic state in the ith layer, for the generalized Phillips model, $U_i=-\partial\Psi_i/\partial y$, where $i=1,2,3$ are constants. (ψ_i, q_i) is a disturbance superimposed on the steady basic state,

$$\Phi_i=-U_i y+\psi_i,\ P_i=Q_i+q_i \tag{6}$$

and

$$q_i=\nabla^2\psi_i-d_i^{-1}\sum_{j=1}^{3}T_{ij}\psi_j,\qquad i=1,2,3. \tag{7}$$

3. Linear theory (Normal mode method)

Substituting (6) and (7) into (1)–(2) and linearizing the system, we have

$$\frac{\partial q_i}{\partial t}+U_i\frac{\partial q_i}{\partial x}+\frac{\partial\psi_i}{\partial x}\frac{\partial Q_i}{\partial y}=0,\qquad i=1,2,3. \tag{8}$$

where

$$q_1=\nabla^2\psi_1-F_1((1+\alpha)\psi_1-\psi_2), \tag{9a}$$

$$q_2=\nabla^2\psi_2-F_2(2\psi_2-\psi_1-\psi_3), \tag{9b}$$

$$q_3=\nabla^2\psi_3-F_3(\psi_3-\psi_2). \tag{9c}$$

and

$$\frac{\partial Q_1}{\partial y} = \beta + F_1((1+\alpha)U_1 - U_2), \quad (10a)$$

$$\frac{\partial Q_2}{\partial y} = \beta + F_2(2U_2 - U_1 - U_3), \quad (10b)$$

$$\frac{\partial Q_3}{\partial y} = \beta + F_3(U_3 - U_2), \quad (10c)$$

where $F_i = d_i^{-1} f_0^2 g_1^{-1} = d_i^{-1} f_0^2 g_2^{-1}$. The boundary conditions of the disturbance (ψ_i, q_i) are

$$\frac{\partial \psi_i}{\partial x} = 0, \quad y = \pm Y, \quad i = 1,2,3, \quad (11a)$$

$$\frac{\partial}{\partial t}\int_{-\pi L}^{\pi L} \frac{\partial \psi_i}{\partial y} dx = 0, \quad y = \pm Y, \quad i = 1,2,3. \quad (11b)$$

Thus, we can investigate the normal mode solutions to (8) and (11) in the form

$$\psi_i = A_i \cos(l_j y) e^{ik(x-ct)}, \quad (12)$$

where A_i, $i = 1,2,3$, is the disturbance amplitude (constant) in the ith layer, and $l_j = (j + \frac{1}{2})\pi / Y$, $j = 0,1\cdots$.

Substituting (12) into (8) yields three algebraic equations for A_i, $i = 1,2,3$, i.e.,

$$A_1\{(c - U_1)(K^2 + F_1(1+\alpha)) + \beta + F_1[(1+\alpha)U_1 - U_2]\} - A_2(c - U_1)F_1 = 0, \quad (13a)$$

$$A_2[(c - U_2)(K^2 + 2F_2) + \beta + F_2(2U_2 - U_1 - U_3)] - A_1(c - U_2)F_2 - A_3(c - U_2)F_2 = 0, \quad (13b)$$

$$A_3[(c - U_3)(K^2 + F_3) + \beta + F_3(U_3 - U_2)] - A_2(c - U_3)F_3 = 0, \quad (13c)$$

where $K^2 = k^2 + l^2$ is the total wavenumber.

For simplification we suppose that $d_1 = d_2 = d_3$, i.e., $F_1 = F_2 = F_3 = F$, and consider the steady basic state as follows (Held and O'Brien, 1987):

$$U_1 = V, \quad U_2 = \frac{V - \chi}{2}, \quad U_3 = 0, \quad (14)$$

where V is a positive constant; χ is the curvature in the vertical profile of the horizontal velocity.

$$\chi = (U_3 - U_2) - (U_2 - U_1). \quad (15)$$

when $\chi = V$ (or $\chi = -V$), the shear is concentrated in top (bottom) two layers; when $\chi = 0$, the top shear is equal to the bottom shear.

Nontrivial solutions for A_1, A_2 and A_3 are possible only if the determinant of the coefficients in (13) vanishes. This condition leads directly to a cubic equation for c:

$$A_5 \cdot c^3 + A_6 \cdot c^2 + A_7 \cdot c + A_8 = 0, \quad (16)$$

where

$$A_5 = \alpha + 3(1+\alpha)\mu + \mu^3 + (\alpha + 4)\mu^2, \quad (17a)$$

$$A_6 = -\frac{3}{2}\alpha V + 3\alpha B + 3B + \frac{1}{2}\alpha\chi + (-\frac{9}{2}V + 2\alpha B - 2\alpha V + 8B + \alpha\chi + \frac{5}{2}\chi)\mu$$

$$+ (\frac{1}{2}\chi - \frac{3}{2}V)\mu^3 + (\frac{1}{2}\alpha\chi - \frac{1}{2}\alpha V + 2\chi - 6V + 3B)\mu^2, \tag{17b}$$

$$A_7 = 4B^2 - \frac{3}{2}\alpha VB - 3BV + 2B\chi + \frac{1}{2}\alpha B\chi + \frac{1}{2}\alpha V^2 - \frac{1}{2}\alpha V\chi + \alpha B^2 + \frac{3}{4}\chi^2 - \frac{1}{2}\alpha VB$$
$$+ (\frac{11}{4}V^2 - 8BV + 3B\chi + \frac{1}{4}\alpha\chi^2 + \frac{1}{2}\alpha B\chi + \frac{1}{4}\alpha V^2 - \frac{1}{2}\alpha V\chi + 3B^2 - \frac{5}{2}V\chi)\mu$$
$$+ (-\frac{1}{2}V\chi + \frac{1}{2}V^3)\mu^3 + (B\chi + \frac{5}{2}V^2 + \frac{1}{2}\chi - 2V\chi - 3BV)\mu^2, \tag{17c}$$

$$A_8 = -BV\chi - \frac{3}{4}BV^2 - 2VB^2 + \frac{1}{4}B\chi^2 + B^3 + B^2\chi + (\frac{1}{8}\chi^3 - \frac{3}{2}VB^2$$
$$- \frac{5}{8}V^3 - \frac{3}{2}BV\chi - \frac{3}{8}V\chi^2 + \frac{1}{2}B\chi^2 + 2BV^2 + \frac{7}{8}V^2\chi + \frac{1}{2}B^2\chi)\mu$$
$$+ (-\frac{1}{2}BV\chi - \frac{1}{4}V\chi^2 - \frac{1}{4}V^3 + \frac{1}{2}BV^2 + \frac{1}{2}V^2\chi)\mu^2, \tag{17d}$$

where $B = \beta / F$ and $\mu = K^2 / F$. We define

$$p = \frac{A_7}{3A_5} - (\frac{A_6}{3A_5})^3, \tag{18a}$$

$$r = \frac{A_7}{2A_5} \cdot \frac{A_6}{3A_5} - (\frac{A_6}{3A_5})^3 - \frac{A_8}{2A_5}. \tag{18b}$$

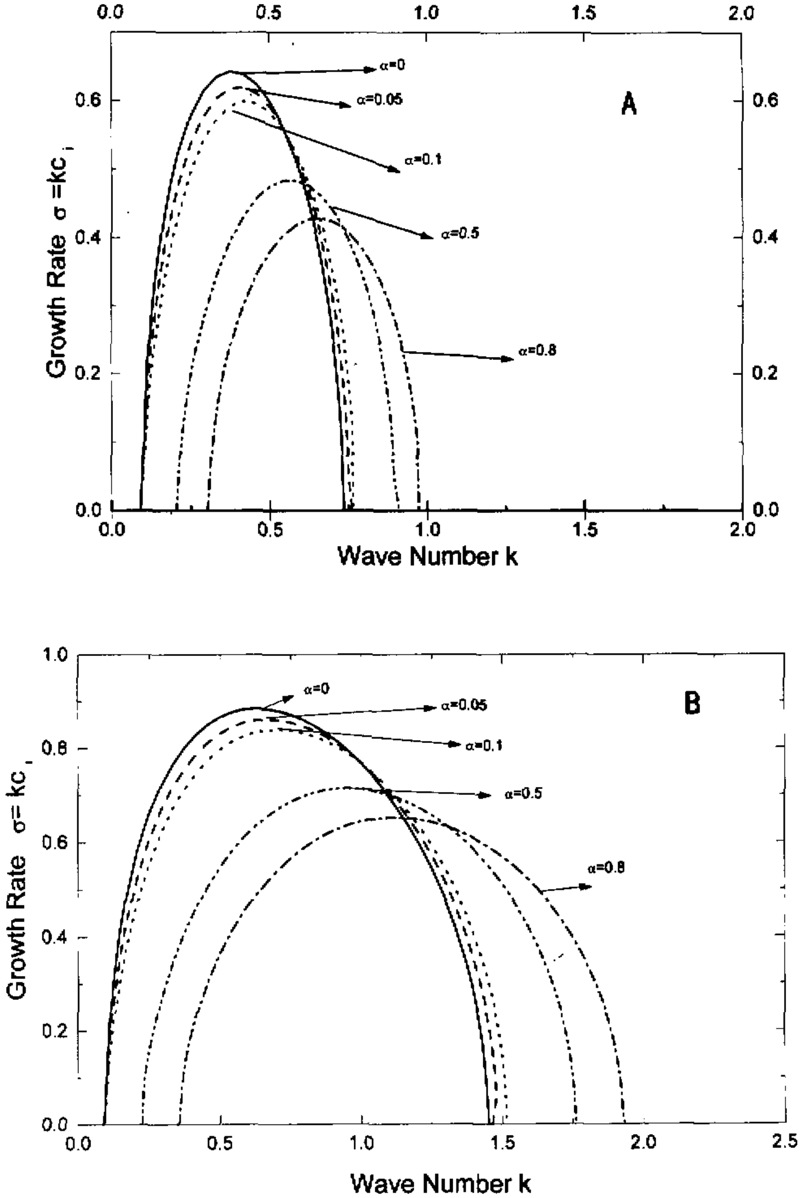
It follows that the sufficient and necessary condition for instability $c \neq 0$ is as follows:

$$\Delta_l = p^3 + r^2 > 0. \tag{19}$$

First, we can derive the non–dimensional expression of (19), which is the same as the inequality (19) in form. Then we draw out the growth rate curves for different parameters, and discuss the effects of the free surface parameter, the curvature in the vertical profile of the horizontal velocity and the parameter β on the linear baroclinic instability.

Figure 1 shows the growth rate curves for $B = 0$, $V = 1$, $\chi = 0, 0.5, 1.0$, and free surface parameter $\alpha = 0$, 0.05, 0.1, 0.5, 0.8 respectively, where $l = \pi / 2$. Figure 1 gives the greatest growth rate and the unstable domain in the above cases. It is shown that there exist not only the short–wave cutoff but also the long–wave cutoff. We can see that when the free parameter increases, which is similar to the two–layer case (e.g., Li and Mu, 1996), the greatest growth rate decreases, the unstable domain moves in the direction of large wavenumber (i.e., short wave direction), and the most unstable wavelength decreases. Fig. 2 is similar to Fig. 1, the parameter B is taken as 0.5, and the results are also similar to that in Fig. 1. This phenomenon shows that the free surface parameter restrains the instability.

Figure 3 shows the growth rate curves for $B = 0$, $V = 1$, $\alpha = 0$ and 0.1, and $\chi = 0$, ± 0.2, ± 0.5, ± 0.8, ± 1.0, respectively. From Fig. 3 we can see that as the absolute value of χ increases, i.e., as the shear concentrates in the top (or bottom) layer, the greatest growth rate increases, the unstable domain extends, and the most unstable wavelength decreases. It means that the larger the curvature is, the more unstable the basic state is. For Fig.3a, $\alpha = 0$, i.e., both the top and bottom surfaces are rigid, the growth rate curves of $\chi = 0.2$, 0.5, 0.8, 1.0 and those curves of $\chi = -0.2$, -0.5, -0.8, -1.0 are the same, respectively, due to the symmetry of fluid. For Fig. 3b, $\alpha = 0.1$, i.e., the top surface is free, the curves of $\chi = 0.2$, 0.5, 0.8, 1.0 and those of $\chi = -0.2$, -0.5, -0.8, -1.0 are different. Figure 4 is similar to Fig. 3, but the parameters are $\alpha = 0$, $V = 1$, $B = 0.5$, 1.0 and $\chi = 0$, ± 0.2, ± 0.5, ± 0.8, ± 1.0, respectively.


A
B
α=0
α=0.05
α=0.1
α=0.5
α=0.8
Growth Rate σ =kc$_i$
Wave Number k

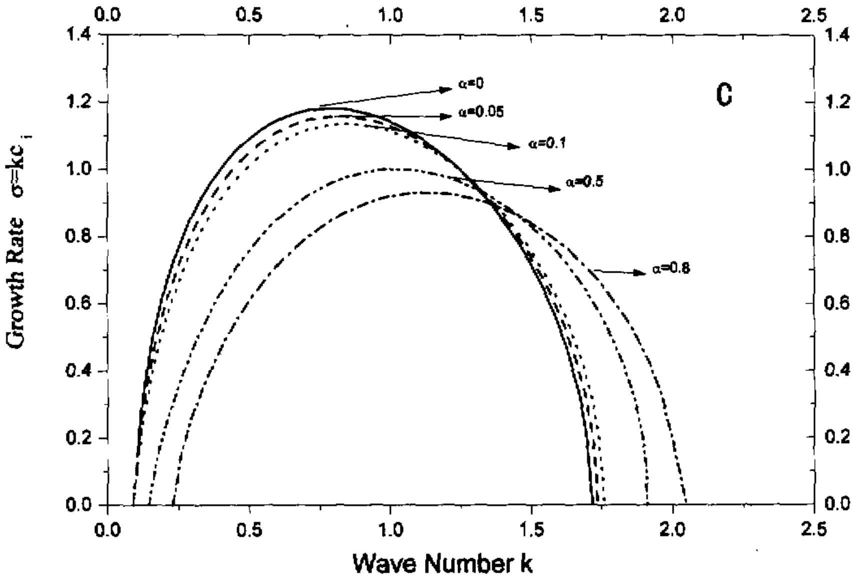


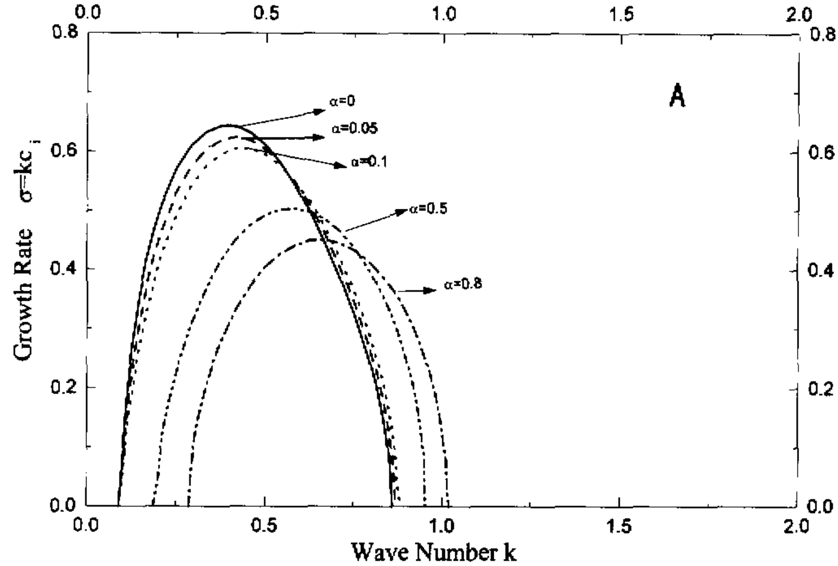
Fig. 1. The growth rate curves in the case of the parameters $B=0$, $V=1.0$, $\alpha=0$, 0.05, 0.1, 0.5, 0.8, respectively, when (a) $\chi=0$, (b) $\chi=0.5$, and (c) $\chi=1.0$.

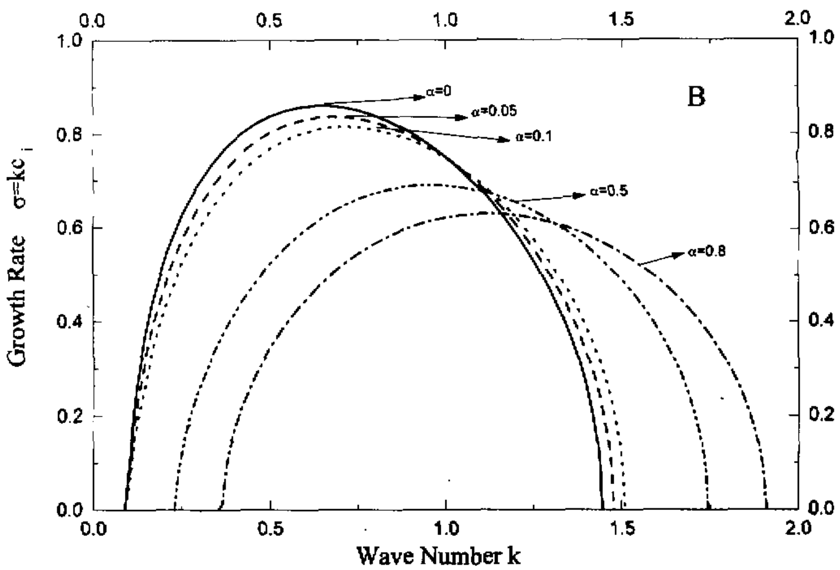
The results are also similar to those in Fig. 3a. It is shown that the uneven distribution of PV flux in the vertical is one of the important factors in the baroclinic instability; the more uneven the distribution of PV flux in the vertical is, the more unstable the basic state is.

Figure 5 shows the growth rate curves for different β parameters, i.e., $B=0.0$, 0.5, 1.0, 2.0, respectively. For Fig. 5a, $V=1$, $\chi=0$, and $\alpha=0$, i.e., the top and bottom surfaces are rigid and the top shear is equal to the bottom one. From Fig. 5a we can see that as parameter β increases, the greatest growth rate becomes similar to that in the case of $B=0$, the unstable domain moves in the direction of large wavenumber (i.e., short wave direction) and the most unstable wavelength decreases. In the case of Fig. 5b, $V=1$, $\chi=0.2$, $\alpha=0.1$, and $B=0$, 0.5, 1.0, 2.0, which means that the top surface is free and the top shear is larger than the bottom one. It is shown that when β parameter increases, the greatest growth rate decreases, the unstable domain contracts, and the most unstable wavelength increases. It is also shown that the uneven distribution of PV flux in the vertical is a very important factor for the baroclinic instability; β parameter (B) restrains the instability of basic state when the distribution of PV flux in the vertical is uneven, while the restraint role of parameter β is not evident when the distribution of PV flux in the vertical is even.

4. Nonlinear stability

Assume that (Ψ_i, Q_i) is the meridionally symmetric basic state and there exist a constant γ and functions $\Psi_i^r(\cdot)$ such that


A
α=0
α=0.05
α=0.1
α=0.5
α=0.8
Growth Rate σ=kc_i
Wave Number k
0.0 0.5 1.0 1.5 2.0
0.0 0.2 0.4 0.6 0.8


B
α=0
α=0.05
α=0.1
α=0.5
α=0.8
Growth Rate σ=kc_i
Wave Number k
0.0 0.5 1.0 1.5 2.0
0.0 0.2 0.4 0.6 0.8 1.0

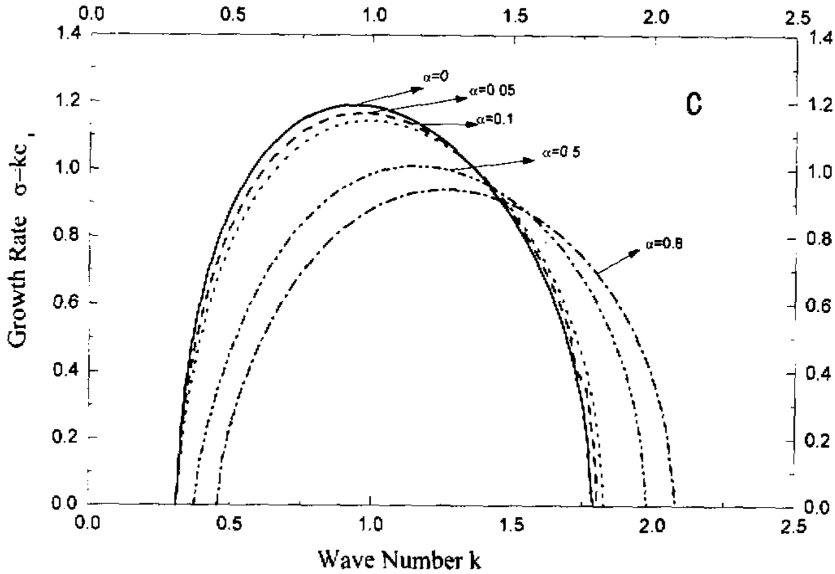


Fig. 2. As in Fig. 1 except for $B=0.5$.

$$\Psi_i + \gamma y = \Psi_i^{\gamma}(Q_i), \quad i = 1,2,3. \tag{20}$$

And suppose that there exist constants C_{1i} and C_{2i}, $i=1,2,3$, such that

$$0 < C_{1i} \leqslant \frac{d\Psi_i^{\gamma}}{dQ_i} \leqslant C_{2i} < \infty, \quad i = 1,2,3. \tag{21}$$

Using the results of of Mu et al. (1994), the basic state is nonlinearly stable if and only if

$$M_{11} = C_{11} - \frac{\lambda^2 + 3F\lambda + F^2}{R} > 0, \tag{22a}$$

$$M_{22} = C_{12} - \frac{(\lambda + F + F\alpha)(\lambda + F)}{R} > 0, \tag{22b}$$

$$M_{33} = C_{13} - \frac{\lambda^2 + 3F\lambda + F^2 + F\alpha\lambda + 2aF^2}{R} > 0, \tag{22c}$$

$$(C_{11} - \frac{\lambda^2 + 3F\lambda + F^2}{R})(C_{12} - \frac{(\lambda + F + F\alpha)(\lambda + F)}{R}) > \frac{F^2(\lambda + F)^2}{R^2}, \tag{22d}$$

$$(C_{12} - \frac{(\lambda + F + F\alpha)(\lambda + F)}{R})(C_{13} - \frac{\lambda^2 + 3F\lambda + F^2 + F\alpha\lambda + 2aF^2}{R}) > \frac{(\lambda + F(1+\alpha))^2 F^2}{R^2}, \tag{22e}$$

$$(C_{13} - \frac{\lambda^2 + 3F\lambda + F^2 + F\alpha\lambda + 2aF^2}{R})(C_{11} - \frac{\lambda^2 + 3F\lambda + F^2}{R}) > \frac{F^4}{R^2} \tag{22f}$$

$$\det M > 0. \tag{22g}$$

where $R = \lambda^3 + (4+\alpha)F\lambda^2 + (4+3\alpha)F^2\lambda + \alpha F^3$.

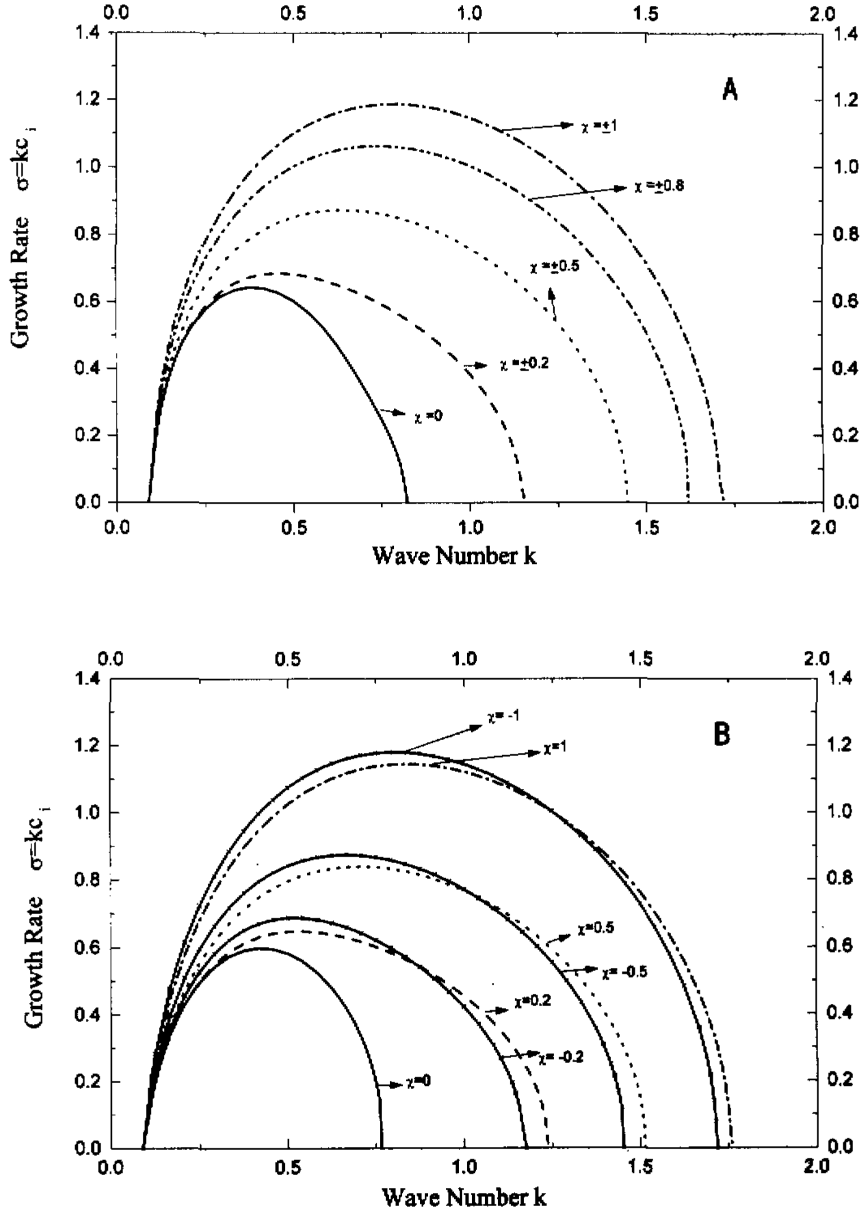
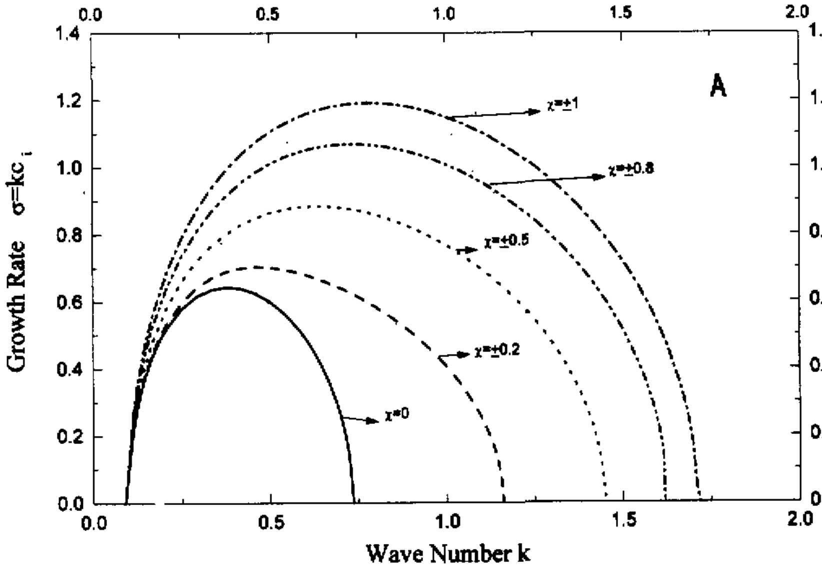
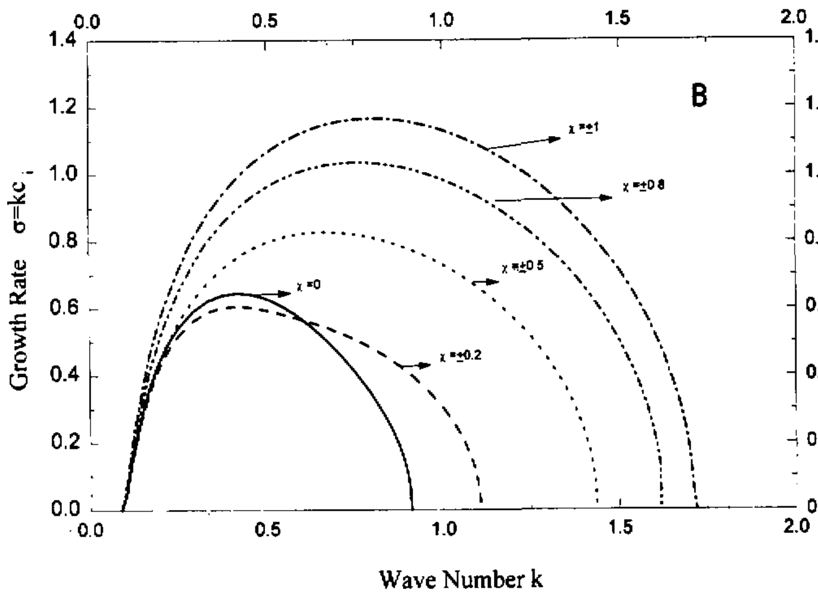


Fig. 3. The growth rate curves in the case of the parameters $B=0$, $V=1.0$, $\chi=0$, ± 0.2, ± 0.5, ± 0.8, ± 1.0, respectively, when (a) $\alpha=0$ and (b) $\alpha=0.1$.


A
Growth Rate σ=kc_i
Wave Number k
χ=±1
χ=±0.8
χ=±0.5
χ=±0.2
χ=0


B
Growth Rate σ=kc_i
Wave Number k
χ=±1
χ=±0.8
χ=±0.5
χ=0
χ=±0.2

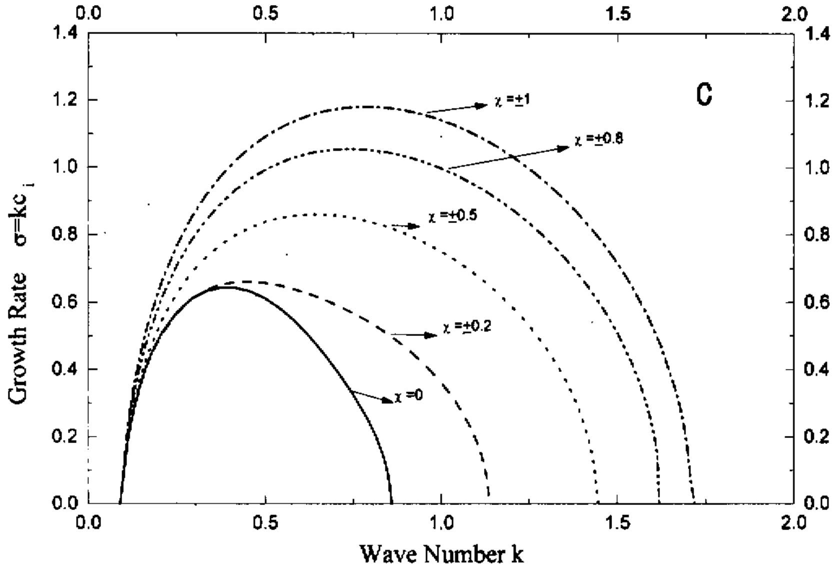


Fig. 4. The growth rate curves in the case of the parameters $\alpha=0$, $V=1.0$, $\chi=0$, ± 0.2, ± 0.5, ± 0.8, ± 1.0, respectively, when (a) $B=0.5$, and (b) $B=1.0$.

For the three-layer generalized Phillips' model, (20) can be written as

$$\Psi_1' = \Psi_1 + \gamma y = (\gamma - V)y, \tag{23a}$$

$$\Psi_2' = \Psi_2 + \gamma y = \left(\gamma - \frac{V-\chi}{2}\right)y, \tag{23b}$$

$$\Psi_3' = \Psi_3 + \gamma y = \gamma y, \tag{23c}$$

and

$$Q_1 = \left(\beta + F\left(\alpha V + \frac{V+\chi}{2}\right)\right)y + f_0, \tag{24a}$$

$$Q_2 = (\beta - F\chi)y + f_0, \tag{24b}$$

$$Q_3 = \left(\beta - F\frac{V-\chi}{2}\right)y + f_0. \tag{24c}$$

Now the problem is reduced to how to choose the parameter γ to establish the nonlinear stability criteria as sharp as possible, following Mu et al. (1994). To this end, we can first choose $C_{1i}, C_{2i}, i=1,2,3$ by (21), (23) and (24), such that (22) holds. According to the symbol of $dQ_1/dy, dQ_2/dy, dQ_3/dy$, this problem can be investigated in the following three cases:

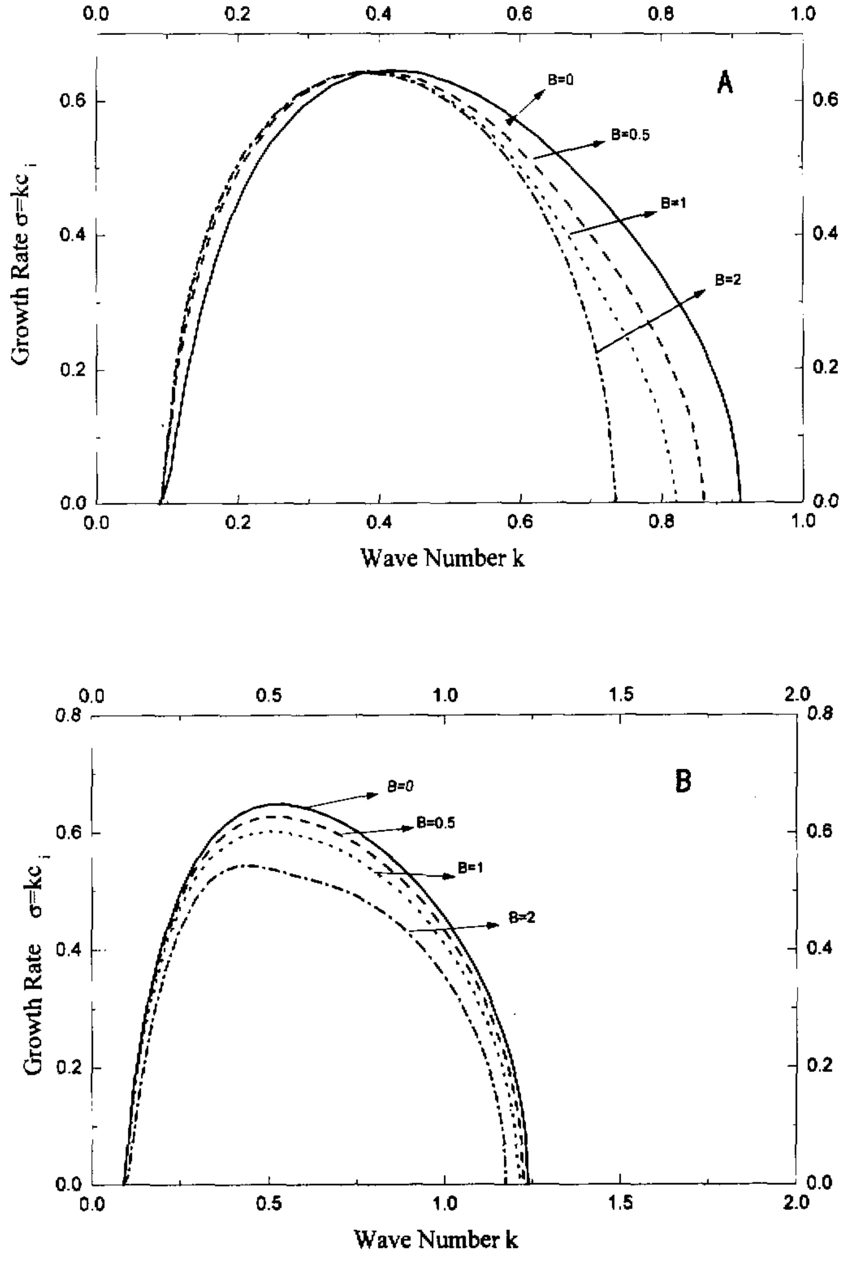


Fig. 5. The growth rate curves in the case of the parameters $V=1.0$, $B=0$, 0.5, 1.0, 2.0, respectively, when (a) $\chi=0$, $\alpha=0$ and (b) $\chi=0.2$, $\alpha=0.1$.

Case 1. $(\beta+F(\alpha V+\frac{V+\chi}{2}))(\beta-F\chi)(\beta-F\frac{V-\chi}{2})=0.$

In this case, when one of $dQ_1/dy, dQ_2/dy, dQ_3/dy$ vanishes, we can obtain the nonlinear criterion as follows:

(i) $\beta+F(\alpha V+\frac{V+\chi}{2})=0$, i.e., $\chi=-2(\frac{\beta}{F}+\alpha V+\frac{V}{2})$.

We choose $\gamma=U_1=V$, then the basic state satisfies (21) when C_{1i}, $i=1,2,3$, satisfies

$$C_{11}=c,\quad C_{12}=-\frac{\gamma-\frac{V-\chi}{2}}{\beta-F\chi}=\frac{V+\chi}{F(2\alpha V+V+3\chi)},\tag{25a,b}$$

$$C_{13}=-\frac{\chi}{\beta-F\frac{V-\chi}{2}}=\frac{1}{F(1+\alpha)},\tag{25c}$$

where c is an arbitrary constant. Substituting (25b,c) into (22a,b,c), we have

$$\frac{V+\chi}{F((2\alpha+1)V+3\chi)}-\frac{(\chi+F(1+\alpha))(\gamma+F)}{R}>0,\tag{26}$$

$$\frac{1}{F(1+\alpha)}-\frac{\lambda^2(3F+\alpha F)+(2\alpha+1)F^2}{R}>0,\tag{27}$$

$$(\frac{V+\chi}{F((2\alpha+1)V+3\chi)}-\frac{(\lambda+F(1+\alpha))(\lambda+F)}{R})\times$$
$$(\frac{1}{F(1+\alpha)}-\frac{\lambda^2(3F+\alpha F)\lambda+(2\alpha+1)F^2}{R})>\frac{(\lambda+F(1+\alpha))^2F^2}{R^2}.\tag{28}$$

When c is chosen to be sufficiently large, (22a,d,f,g) may be satisfied. Hence, (26)–(28) are sufficient conditions for the nonlinear stability of the basic state.

(ii) $\beta-F\chi=0$, i.e., $\chi=\beta/F$.

Similarly to (i), we choose $\gamma=(V-\chi)/2$, and

$$C_{11}=-\frac{\gamma-V}{\beta+F(\alpha V+\frac{V+\chi}{2})}=\frac{V+\chi}{F(2\alpha V+V+3\chi)},\tag{29a}$$

$$C_{12}=c,\quad C_{13}=-\frac{\gamma}{\beta-F\frac{V-\chi}{2}}=\frac{V-\chi}{F(V-3\chi)}.\tag{29b,c}$$

Then, sufficient conditions for the nonlinear stability of the basic state (Ψ_i, Q_i) are

$$\frac{V+\chi}{F(2\alpha V+V+3\chi)}-\frac{\lambda^2+3F\lambda+F^2}{R}>0,\tag{30a}$$

$$\frac{V-\chi}{F(V-3\chi)}-\frac{\lambda^2(3F+\alpha F)\lambda+(2\alpha+1)F^2}{R}>0,\tag{30b}$$

$$(\frac{V-\chi}{F(V-3\chi)}-\frac{\lambda^2(3F+\alpha F)\lambda+(2\alpha+1)F^2}{R})(\frac{V+\chi}{F(2\alpha V+V+3\chi)}-\frac{\lambda^2+3F\lambda+F^2}{R})>\frac{F^4}{R^2}.\tag{30c}$$

(iii) $\beta-F\frac{V-\chi}{2}=0$, i.e., $\chi=-\frac{2\beta}{F}+V$.

Similarly to (iii), we choose $\gamma=0$,

$$C_{11}=-\frac{\gamma-V}{\beta+F(\alpha V+\frac{V+\chi}{2})}=\frac{1}{F(1+\alpha)}, \tag{31a}$$

$$C_{12}=-\frac{\gamma-\frac{(V-\chi)}{2}}{\beta-F\chi}=\frac{V-\chi}{F(V-3\chi)},\quad C_{13}=c. \tag{31b,c}$$

Sufficient conditions for the nonlinear stability of the basic state are

$$\frac{1}{F(1+\alpha)}-\frac{\lambda^2+3F\lambda+F^2}{R}>0, \tag{32a}$$

$$\frac{V-\chi}{F(V-2\chi)}-\frac{(\lambda+F(1+\alpha))(\lambda+F)}{R}>0, \tag{32b}$$

$$\left(\frac{1}{F(1+\alpha)}-\frac{\lambda^2+3F\lambda+F^2}{R}\right)\left(\frac{V-\chi}{F(V-2\chi)}-\frac{(\lambda+F(1+\alpha))(\lambda+F)}{R}\right)>\frac{F^2(\lambda+F)^2}{R^2}. \tag{32c}$$

Case 2. $(\beta+F(\alpha V+\frac{V+\chi}{2}))(\beta-F\chi)(\beta-F\frac{V-\chi}{2})>0.$

To obtain the nonlinear stability criterion, we can consider the following cases:

(i) $\beta+F(\alpha V+\frac{V+\chi}{2})>0,\ \beta-F\chi>0,\ \beta-F\frac{V-\chi}{2}>0,$ (33)

thus

$$\gamma-V<0,\quad \gamma-\frac{V-\chi}{2}<0,\quad \gamma<0, \tag{34a}$$

i.e.,

$$\gamma<\min(V,\frac{V-\chi}{2},0). \tag{34b}$$

If we choose $\gamma\to-\infty$, such that

$$C_{11}=-\frac{\gamma-V}{\beta+F(\alpha V+\frac{V+\chi}{2})},$$

$$C_{12}=-\frac{\gamma-(V-\chi)/2}{\beta-F\chi}\quad\text{and}\quad C_{13}=-\frac{\gamma}{\beta-F\frac{V-\chi}{2}}$$

can be chosen to be sufficient large, (22a–g) may be satisfied, i.e.,

$$\max(-2(\frac{\beta}{F}+\alpha V)-V,\ V-\frac{2\beta}{F})<\chi<\frac{\beta}{F} \tag{35}$$

is one sufficient condition for the nonlinear stability.

(ii) $\beta+F(\alpha V+\frac{V+\chi}{2})<0,\quad \beta-F\chi<0,\quad \beta-F\frac{V-\chi}{2}>0.$ (36a)

This requires that

$$\max(\frac{\beta}{F},V-\frac{2\beta}{F})<\chi<-2(\frac{\beta}{F}+\alpha V)-V, \tag{36b}$$

and $V>0$, therefore there does not exist γ that satisfies (36). Then, in this case, there does not exist V that satisfies (21).

(iii) $\beta+F(\alpha V+\frac{V+\chi}{2})<0,\quad \beta-F\chi>0,\quad \beta-F\frac{V-\chi}{2}<0.$ (37)

Similarly to (ii), there does not exist V that satisfies (21).

(iv) $\beta+F(\alpha V+\frac{V+\chi}{2})>0,\quad \beta-F\chi<0,\quad \beta-F\frac{V-\chi}{2}<0.$ (38)

In this case,

$$\gamma-V<0,\quad \gamma-\frac{V-\chi}{2}>0,\quad \gamma>0, \tag{39a}$$

i.e.,

$$\max(0,\frac{V+\chi}{2})<\gamma<V. \tag{39b}$$

From (38), we have

$$\frac{\beta}{F}<\chi<-\frac{2\beta}{F}+V, \tag{40}$$

$$C_{11}=-\frac{\gamma-V}{\beta+F(\alpha V+\frac{V+\chi}{2})},\quad C_{12}=-\frac{\gamma-\frac{V+\chi}{2}}{\beta-F\chi}, \tag{41a,b}$$

$$C_{13}=-\frac{\gamma}{\beta-F\frac{V-\chi}{2}}. \tag{41c}$$

Substituting (41a–c) into (22a–g), we obtain one sufficient condition for the nonlinear stability, which are noted as (22a–g)′.

Case 3. $(\beta+F(\alpha V+\frac{V+\chi}{2}))(\beta-F\chi)(\beta-F\frac{V-\chi}{2})<0.$

Similarly to **Case 2**, we have

(i) $\beta+F(\alpha V+\frac{V+\chi}{2})<0,\quad \beta-F\chi<0,\quad \beta-F\frac{V-\chi}{2}<0,$ (42)

i.e.,

$$\frac{\beta}{F}<\chi<-\frac{2\beta}{F}-2\alpha V-V.$$

Since both α and U are positive, in this case there does not exist χ that satisfies (21).

(ii) $\beta+F(\alpha V+\frac{V+\chi}{2})<0,\quad \beta-F\chi>0,\quad \beta-F\frac{V-\chi}{2}>0,$ (43)

i.e.,

$$V-\frac{2\beta}{F}<\chi<-\frac{2\beta}{F}-2\alpha V-V.$$

In this case there does not exist χ that satisfies (21).

(iii) $\beta+F(\alpha V+\frac{V+\chi}{2})>0,\quad \beta-F\chi<0,\quad \beta-F\frac{V-\chi}{2}>0.$ (44)

Similarly to (ii), in this case there do not exist χ and γ that satisfy (21).

(iv) $\beta+F(\alpha V+\frac{V+\chi}{2})>0,\quad \beta-F\chi<0,\quad \beta-F\frac{V-\chi}{2}<0,$ (45)

i.e.,

$$-\frac{2\beta}{F}-2\alpha V<\chi<\min(\frac{\beta}{F},V-\frac{2\beta}{F}), \tag{46a}$$

and

$$0<\gamma<\min(V,\frac{V+\chi}{2}). \tag{46b}$$

Thus, we choose C_{11}, C_{12}, C_{13} as (41a–c), and substitute them into (22a–g). We gain one sufficient condition for the nonlinear stability, which is noted as (22a–g)′.

In summary, we have established the following criterion:

Criterion 1. The basic state of the three–layer generalized Phillips′ model is nonlinearly stable if one of the following conditions is satisfied:

a) $\chi=-2(\frac{\beta}{F}+\alpha V+\frac{V}{2})$ and (26)–(28) hold;

b) $\chi=\beta/F$ and (30a–c) hold;

c) $\chi=-\frac{2\beta}{F}+V$ and (32a–c) hold;

d) $V-\frac{2\beta}{F}<\chi<\frac{\beta}{F}$;

e) $\frac{\beta}{F}<\chi<-\frac{2\beta}{F}+V$ or $-\frac{2\beta}{F}-2\alpha V-V<\chi<\min(\frac{\beta}{F},V-\frac{2\beta}{F})$ and (22a–g)′ hold.

Now let us consider condition e) in details, (22g)′ can be written as:

$$N_5\cdot\gamma^3+N_6\cdot\gamma^2+N_7\cdot\gamma+N_8>0, \tag{47}$$

where

$$N_5=8\alpha(24\alpha+24)\bar{\lambda}+8\bar{\lambda}^3+(32+8\alpha)\bar{\lambda}^2. \tag{48a}$$

$$N_6=24B-12\alpha V+24\alpha B-4\alpha\chi+(-4\chi+64B-16\alpha V-36V+16\alpha B)\bar{\lambda}$$
$$(-4\chi-12V)\bar{\lambda}^3+(-48V-16\chi-4\alpha\chi-4aV+24B)\bar{\lambda}^2, \tag{48b}$$

$$N_7=-8\alpha\chi^2+4\alpha V\chi+32B^2-8\chi^2+8\alpha B^2-12\alpha\chi B-24VB+4V^2\alpha-12\alpha VB+(-2\alpha\chi^2$$
$$-18\chi^2-4\alpha\chi B-64VB+22V^2-4\alpha VB+4V\chi+24B^2-24\chi B+2V^2\alpha)\bar{\lambda}$$
$$(4V\chi+4V^2)\bar{\lambda}^3+(-8\chi B+16V\chi-24VB+20V^2-4\chi^2)\bar{\lambda}^2, \tag{48c}$$

$$N_8=8B^3-8\chi B^2-14\chi^2B+6V^2B+4V\chi^2-16B^2V-4\chi^3$$
$$+(-\chi^3+9V\chi^2-5V^3-4\chi^2B+16V^2B-3V^2\chi-12B^2V$$
$$-4\chi B^2+12VB\chi)\bar{\lambda}+(-2V^3+2V\chi^2+4VB\chi+4V^2B)\bar{\lambda}^2, \tag{48d}$$

where $\bar{\lambda}=\lambda/F$, $B=\beta/F$. Then, there exists γ satisfying (39b) and (46b), such that (48) hold if and only if

$$\nabla_n=p_n^3+r_n^2<0, \tag{49}$$

or

$$N_5>0,\quad N_8>0. \tag{50}$$

where

$$p_n=\frac{N_7}{3N_5}-(\frac{N_6}{3N_5})^3, \tag{51a}$$

$$r_n = \frac{N_7}{2N_5} \cdot \frac{N_6}{3N_5} - (\frac{N_6}{3N_5})^3 - \frac{N_8}{2N_5}. \tag{51b}$$

Comparing (49) with (19), and letting $V = \beta / F$, we find that (19) is exactly the opposite of (49) if we set $\mu = \bar{\lambda}$ in (19). But when we choose $V = 2\beta / F$, we cannot obtain the above result. In the following, we will explain the phenomenon using the theory of explosive resonant interaction (ERI).

In general, when the nonlinear stability conditions and the linear (normal model) instability conditions do not coincide in parameter space, two explanations can be possible: the first is that the nonlinear stability conditions, being only sufficient, could still be improved; the second is that there exist the nonlinear mechanisms capable of amplifying infinitesimal initial disturbances. The latter explanation should be important and non–surprising: it is well known that the stability of a linearized system does not imply the stability of original nonlinear system, even to arbitrarily small disturbances. Cairns (1979) and Craik and Adam (1979) showed that a resonant triad of gravity waves could extract energy from the basic flow, so the three triad members grew simultaneously. This process is called explosive resonant interaction (ERI).

Vanneste (1995) discussed the explosive resonant interaction of baroclinic Rossby waves. He obtained the sufficient and necessary conditions of existence of ERI for the three–layer quasi–geostrophic model with top and bottom surfaces being rigid, and revealed the relation between the nonlinear stability and the linear stability of that model: the system is nonlinearly unstable even when it is linearly stable, and ERI plays a leading role in the instability. The sufficient and necessary conditions of existence of ERI are:

$$S_1 > \frac{\beta}{F} \quad \text{and} \quad 0 < S_2 < \min(\frac{\beta}{F}, S_1 - \frac{\beta}{F}), \tag{52a}$$

or

$$-\frac{\beta}{F} < S_1 < 0 \quad \text{and} \quad S_2 < S_1 - \frac{\beta}{F}, \tag{52b}$$

where $S_1 = U_1 - U_2$, $S_2 = U_2 - U_3$.

For the generalized Phillips' model discussed in this paper, we can easily gain the sufficient and necessary conditions of existence of ERI for this model following the approach of Vanneste (1995):

$$S_1 > \frac{\beta}{F} \quad \text{and} \quad 0 < S_2 < \min(\frac{\beta}{F}, \ S_1 - \frac{\beta}{F}), \tag{53a}$$

or

$$-\alpha V - \frac{\beta}{F} < S_1 < 0 \quad \text{and} \quad S_2 < S_1 - \frac{\beta}{F}. \tag{53b}$$

From (14), condition (53) can be written as

$$\max(\frac{2\beta}{F} - V, V - \frac{2\beta}{F}, \frac{\beta}{F}) < \chi < V. \tag{54}$$

Thus, (54) holds if and only if

$$V > \frac{\beta}{F}. \tag{55}$$

That is to say, when $V \leqslant \beta / F$, there does not exist ERI, and the nonlinear stability criterion is the same as the linear one in form; when $V > \beta / F$, there exists ERI, and ERI plays a leading role in the instability. More precisely, when a basic state is linearly stable, it may be nonlinearly unstable due to the role of ERI, i.e., the nonlinear mechanisms amplify the infinitesimal initial disturbances. In this case the nonlinear stability criterion is different from the linear one. ERI gives an explanation of the relationship between (19) and (49).

5. Summary

By using the normal mode method, the linear instability for the three–layer generalized Phillips' model has been studied, and the linear instability criterion is obtained. The influences of the free parameter, β parameter and curvature in the vertical profile of the horizontal velocity on instability are discussed by use of the growth rate curves. It is shown that (i) when the free surface parameter increases, the greatest growth rate decreases, and the unstable domain moves in the direction of small wavenumber (i.e., long wave direction). The free parameter has a role of restraining the instability. (ii) When the absolute value of χ increases, (i.e., the shear concentrates in the top (or bottom) layer), the greatest growth rate increases, and the unstable domain extends. It is also shown that the uneven distribution of PV flux in the vertical is one of the important reasons for the baroclinic instability, i.e., the more uneven the distribution of PV flux in the vertical is, the more unstable the basic state is. (iii) When the β parameter increases, in the case that the surfaces are rigid and the distribution of vertical PV flux is equal, the greatest growth rate is similar to the case of $\beta = 0$, only the unstable domain moves in the direction of large wavenumber (i.e., short wave direction); but in the case that the top surface is free and the distribution is unequal, the greatest growth rate decreases and the unstable domain reduces. This shows that β parameter restrains the instability of basic state in the case of the uneven distribution of PV flux in the vertical.

For the three–layer case, there exists ERI, which gave an explanation for the difference between the linear instability criterion and the nonlinear one; while there does not exist this phenomenon for the two–layer case (Li and Mu, 1996).

It is of great importance to apply these theoretical results to the practical problem. Many scientists have been working in this respect. We have discussed the persistent anomalous flow over the subtropical western Pacific by using the linear barotropic instability theory in the recent study, which deserves further investigation by using the nonlinear theory of baroclinic instability.

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