The 3D Spiral Structure Pattern in the Atmosphere①

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ABSTRACT

The steady analytical solution of the perturbation equations describing the mesoscale vortex in the atmosphere is obtained. By analyzing the 3D wind field, the 3D spiral structure in physical space for unstable stratification is found. In many respects such as the 3D distribution of pressure and vorticity fields, the mesoscale spiral vortex is very similar to typically real cyclones.

Key words: Spiral structure, Cyclones, 3D dynamical system

1. Introduction

Spiral structures of vortices in different scales have been widely studied. From the synoptic map we can see that the trough line orientation of Rossby wave changes with latitude. A barotropic model in the polar coordinate system was established for the explanation of this observational fact (Chao and Ye, 1977). In the framework of the wave propagation theory, the spiral structure would be organized by the phase isoline of Rossby wave and be responsible for the trough line tilting and maintenance of general circulation. In addition, although the mesoscale tropical cyclone is a complex system of interacting physical processes and multi–scale motions, if neglecting its important characteristics such as the persistent updraft and precipitation in the eye wall clouds, it may be thought of as an axisymmetric vortex in a steady state. Its spiral structure can be generated due to the action of internal gravity waves and the outward spiraling airflow in a low–level layer above the ocean surface (Liu and Yang, 1980).

Exact solutions of the Navier–Stokes equations for a three–dimensional (3D) vortex have been found out (Burgers, 1940; Burgers, 1948; Sullivan, 1959), of greater attracting is Sullivan's two–cell vortex solution because the flow not only spirals in toward the axis and out along it, but also has a region of reverse flow near the axis. In this paper, we elaborate a theoretical atmospheric model of 3D spiral structure of mesoscale vortices in unstable atmospheric stratification.

2. Basic equations

In the local coordinate system (x,y,z), the Boussinesq–Navier–Stokes equations for vorticity ζ, divergence D, vertical velocity ω, mass conservation and thermodynamics may be

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written in the following non-dimensional form:

$$\frac{\partial \zeta}{\partial t}+\frac{D}{Ro}=-\frac{\zeta}{Re}\ , \tag{1}$$

$$\frac{\partial D}{\partial t}-\frac{\zeta}{Ro}=-\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)-\frac{D}{Re}\ , \tag{2}$$

$$\frac{\partial w}{\partial t}=-\frac{\partial p}{\partial z}+\theta-\frac{w}{Re}\ , \tag{3}$$

$$D+\frac{\partial w}{\partial z}=0\ , \tag{4}$$

$$\frac{\partial \theta}{\partial t}+Riw=-\frac{\theta}{PrRe}\ , \tag{5}$$

where Ro, Re, Ri, Pr are the Rossby, Reynolds, Richardson and Prandtl numbers, respectively. p and θ are the non-dimensional perturbation pressure and potential temperature, respectively.

In the case of steady state, Equations (1)–(5) are reduced to

$$\begin{cases} \dfrac{D}{Ro}=-\dfrac{\zeta}{Re}\ , \\ -\dfrac{\zeta}{Re}=-\left(\dfrac{\partial^2 p}{\partial x^2}+\dfrac{\partial^2 p}{\partial y^2}\right)-\dfrac{D}{Re}\ , \\ 0=-\dfrac{\partial p}{\partial z}+\theta-\dfrac{w}{Re}\ , \\ D+\dfrac{\partial w}{\partial z}=0\ , \\ Riw=-\dfrac{\theta}{PrRe}\ . \end{cases} \tag{6}$$

The vertical motion equation is obtained by eliminating perturbation variables ζ, D, θ and P in the form

$$(T_a+1)\frac{\partial^2 w}{\partial z^2}-(Ra-1)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0\ , \tag{7}$$

where $T_a=\dfrac{Re^2}{Ro^2}$, $Ra=-PrRe^2Ri$ are the Taylor and Rayleigh numbers, respectively. We assume that

$$w=W(z)\cos\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}y\right)\ . \tag{8}$$

Then Eq.(7) becomes the following eigenvalue problem:

$$\frac{d^2 W}{dz^2}+\frac{\pi^2}{4}\frac{Ra-1}{T_a+1}\overline{w}=0 \tag{9}$$

with the boundary conditions $W=0$ at $z=0$ and 1. Equation (9), together with the boundary condition, can construct an eigenvalue problem, so the eigenfunction is $\sin(n\pi z)$ and the eigenvalues satisfy

$$\frac{\pi^2}{4}\frac{Ra-1}{T_a+1}=(n\pi)^2\ , \qquad (n=1,2,3,\cdots)\ . \tag{10}$$

3. The 3D autonomous dynamical system

Taking $n=1$, and substituting Eq.(8) into Eq.(6), we obtain

$$\begin{cases} D=-\pi\cos\pi z\cos\frac{\pi}{2}x\cos\frac{\pi}{2}y\ , \\ \zeta=\frac{Ro}{Re}\pi\cos\pi z\frac{\pi}{2}x\cos\frac{\pi}{2}y\ , \\ \theta=-PrReRi\sin\pi z\cos\frac{\pi}{2}x\cos\frac{\pi}{2}y\ , \\ p=\frac{1}{\pi}\left(PrReRi+\frac{1}{Re}\right)\cos\pi z\cos\frac{\pi}{2}x\cos\frac{\pi}{2}y\ . \end{cases} \tag{11}$$

On the basis of vorticity ζ and divergence D, we can obtain the horizontal velocity fields. We finally find that 3D velocity fields are given by

$$\begin{cases} \dot{x}=u=-2w_0\left(\sin(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)+\sqrt{T_a}\cos(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y)\right)\cos(\pi z)\ , \\ \dot{y}=v=-2w_0\left(\cos(\frac{\pi}{2}x)\sin(\frac{\pi}{2}y)-\sqrt{T_a}\sin(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)\right)\cos(\pi z)\ , \\ \dot{z}=w=w_0\cos(\frac{\pi}{2}x)\cos(\frac{\pi}{2}y)\sin(\pi z)\ , \end{cases} \tag{12}$$

where w_0 is a positive constant for unstable stratification. Eqs.(12) form a 3D autonomous nonlinear dynamical system.

4. Qualitative analysis

In the mesoscale synoptic system, the tropical and mid-latitude cyclones are typical vortices. For a warm cyclone in the Northern Hemisphere, the low-level air parcels rotate counterclockwise and spirally converge toward the low-pressure center. The horizontal convergence field in the lower portion is formed. In contrast, the high-level air parcels rotate clockwise and spirally diverge outward the high-pressure center. The horizontal divergence field in upper portion is formed. The region of diverging air is connected with the region of converging air through air parcel ascent so that the mesoscale system creates a 3D spiral structure (Bluestein, 1992). The tropical cyclone possesses analogic characteristics. In short terms, it is a nearly circular, warm-core vortex, occupying the entire height of the troposhere and extending radially a few hundreds of kilometers. The inward spiraling airflow in a thin layer above the ocean surface is accompanied by the radial outflow in an upper layer.

Eqs.(12) can be used to discuss the 3D spiral structure of a mesoscale vortex. It is clearly seen that the autonomous dynamical system has the multiple equilibrium solutions such as $A(0,0,0)$, $B(0,0,1)$, $C(2,0,0)$ and $D(2,0,1)$. For this approach, the equilibrium solutions A and B represent the lower and upper center of the mesoscale vortex, respectively. The eigenvalues

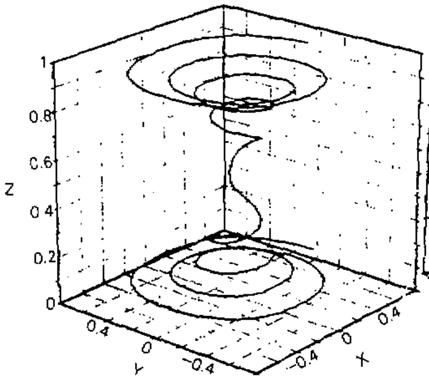

Fig. 1. The heteroclinic orbits existed in the dynamical system (9–11).

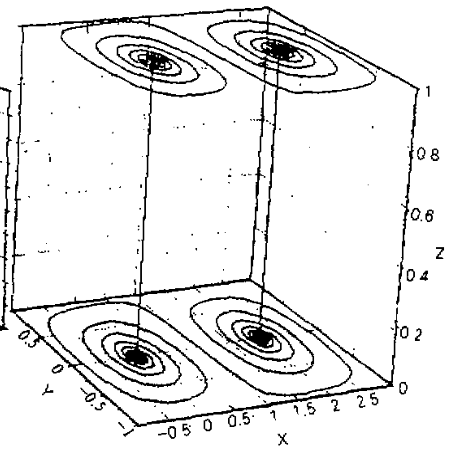

Fig. 2. Trajectory of an air parcel displaying 3D spiral structure for unstable atmospheric stratification.

of Jacobi matrix of linearized expansion near both equilibrium states are $\lambda_1 = \pi w_0$, $\lambda_{2,3} = -\pi w_0(1 \pm i\sqrt{T_a})$ for A and $\lambda_1 = -\pi w_0$, $\lambda_{2,3} = \pi w_0(1 \pm i\sqrt{T_a})$ for B. So both the equilibrium solutions A and B are saddle–focus (Kuzentsov, 1996). Following the similar mathematical calculations, we can confirm that C is similar to B and D similar to A in the aspect of character. The unstable manifold of saddle–focus A (or C) crosses the stable manifold of saddle–focus B (or C). The stable manifold of A (or D) crosses the unstable manifold of B (or C). For heteroclinic connections if can form a heteroclinic cycle as shown in Fig. 1.

The positive real eigenvalue of equilibrium will force the air parcels to go far away from the saddle–focus. On the other hand, the complex eigenvalue with a negative real part will force air parcels to swirl toward the saddle–focus. Fig. 2 shows the trajectory of an air parcel in physical space (x, y, z) for unstable stratification. While the low–level air parcel is spirally converged toward the center A of low A, the air is lifted along the unstable manifold. When the height of air parcel is greater than 0.5, the direction of rotation is changed to be opposite. The air parcel begins to spirally diverge outward the center B of high. Therefore, Fig. 2 can be regarded as a typical or ideal 3D spiral structure, very similar to the pattern of a cyclone vortex in the atmosphere. The dynamical system can provide a powerful tool to analyze theoretically the entire atmopsheric flow patterns of cyclone and anticyclone.

5. Computational results

For understanding 3D structures of the fields of air pressure, temperature and vorticity, Figs. 3 and 4 show their contours at $z = 0.1$ (representing a low altitude) and $z = 0.85$ (representing a high altitude). Notice that the surface cyclone has typical features of low center, warm center and positive vorticity center. The corresponding upper anticyclone possesses typical spatial structure of high center, warm center and negative vorticity. All characteristics of wind, temperature, air pressure and vorticity fields are very similar to the real case of cyclone

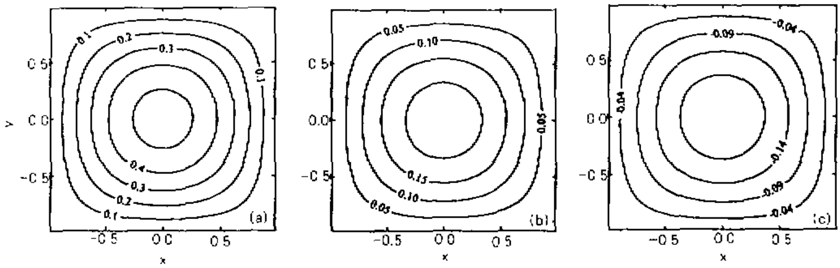

Fig. 3. Distributions of the fields of vorticity (a), temperature (b) and pressure (c) at nondimensional height $z = 0.1$.

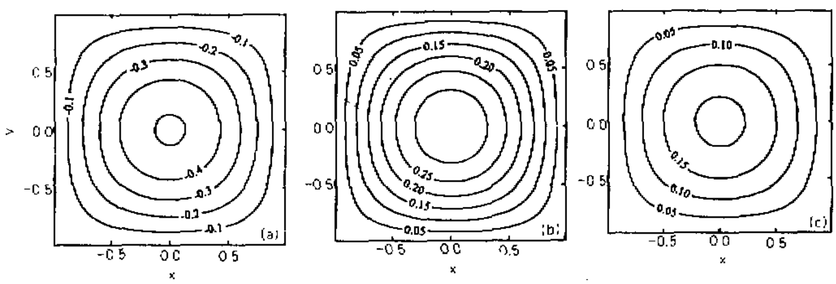

Fig. 4. Same as Fig. 3 but at $z = 0.85$.

and anticyclone. The configuration of upper–level divergence and low–level convergence infers rising air and the possibility of clouds, precipitation. The global features are in good agreement with the midlatitude cyclones or tropical cyclones in the unstable atmospheric stratification (Ahrens, 1994).

In summary, the steady solution of the set of linearized geophysical fluid equations containing the turbulent viscosity and the Coriolis force can construct a nonlinear autonomous system of velocity field in physical space. The 3D wind field makes the air parcel lift spirally in unstable atmospheric environment. The existence of 3D spiral structure in the mesoscale vortex as a result of the mass conservation and effect of earth's rotation, implies that the rotation of the earth and turbulent viscosity of the air play an important role in the 3D spiral structure of the mesoscale vortex.

REFERENCES

Ahrens, C. D., 1994: *Meteorology Today*, West Publishing Company, Minneapolis / St. Paul, New York, Los Angeles, San Francisco, 562 pp.

Burgers, J. M, 1940: Application of a model system to illustrate some point of the statistical theory of free turbulence. *Proc. Acad. Sci. Amsterdam*, **43**, 2–12.

Bluestein, H. B., 1992: *Synoptic–Dynamic Meteorology in Midlatitudes*, Oxford University Press, 431 pp.

Burgers, J. M., 1948: *Mathematical Model Illustrating the Theory of Turbulence. Advances in Applied Mechanics*, **1**, 197–199.

Chandrasekhar, S., 1961: *Hydrodynamic and Hydromagnetic Stability* Oxford University Press, New York, 613 pp.

Chao Jiping and Ye Duzheng, 1977: The spiral planetary waves in the barotropic atmosphere. *Scientia Atmospherica Sinica*, **2**, 81–88 (in Chinese).

Emanuel, K. A., 1994: *Atmospheric Convection*, Oxford University Press, New York, 654 pp.

Kuzentsov, Y. A., 1996: *Elements Applied Bifurcation Theory*, Springer–Verlag, 478 pp.

Liu Shikuo and Yang Dasheng, 1980: The spiral structure of typhoons, *Acta Meteorologica Sinica*, **38**,193–204 (in Chinese).

Sullivan, R. D., 1959: A two–cell vortex solution of the Navier–Stokes equation. *J. Aero/ Space Sci.*, **26**, 767–768.