Diagnostic Equations for the Walker Circulation①

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ABSTRACT

Two linear partial differential equations are derived in spherical–isobaric coordinates for the numerical simulation of the Walker circulation with the assumption that the meridional motion equation remains in gradient balance. One is for the Walker circulation along the individual latitude in the tropical area, the other for the meridionally–averaged Walker circulation over a tropical zone.

Key words: Walker circulation, ENSO, Southern Oscillation

1. Introduction

Large interannual fluctuation in the intensity of the Walker circulation is closely related to the variations of large–scale monsoon circulation and ENSO (El Niño and the Southern Oscillation) which is a recurring climate event bearing blames for many global weather disasters (Schneider, 1996; Suplee, 1999). One of the characters of the Walker circulation is the significant correlation between atmospheric quantities existing over Indonesia and the western Pacific Ocean and the eastern Pacific Ocean. The strongest correlation is the ' southern oscillation index' (Walker, 1923; Troup, 1965) which is a measure of the difference between the standardized sea level pressure anomalies at Tahiti (151°W, 18°S) and Darwin (130°E, 11°S). This index is considered as a firm indicator of variations of El Niño (Trenberth, 1976). From this point of view, the Walker circulation considered in this study is a zonal circulation along 15°S latitude (or those single and meridionally averaged tropical latitudes with relatively strong zonal gradient of divergence) rather than right on the equator (see equation 1).

The motivation of this project is to investigate the evolution mechanisms of these particular Walker circulations through numerical simulation with given forcing functions. This article focuses on the theoretical part of this work: the derivation of diagnostic equations for the Walker circulation along the individual latitude (Section 2) and the meridionally–averaged Walker circulation over a tropical zone (Section 4). Section 3 discusses how to calculate the values of some special forcing factors. The general analytic solution of the equation is given in the appendix.

2. The derivation of a linear diagnostic equation for the walker circulation

To figure out the mechanisms responsible for the variations of the Walker circulation numerically, the best approach is to derive a linear diagnostic equation through assuming that the meridional motion equation remains in gradient balance, that is

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$$u\left(f+u\frac{\tan\varphi}{a}\right)=-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi} \quad . \tag{1}$$

Except for the gradient balance assumption, the rest of equations used in the present study are all in their primitive forms in spherical–isobaric coordinates (λ,φ,p,t), which include the continuity equation

$$\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial(v\cos\varphi)}{\partial\varphi}+\frac{\partial\omega}{\partial p}=0 \ , \tag{2}$$

the zonal motion equation

$$\frac{\partial u}{\partial t}+\frac{v}{a}\frac{\partial u}{\partial\varphi}+\omega\frac{\partial u}{\partial p}+\frac{1}{2a\cos\varphi}\frac{\partial u^2}{\partial\lambda}-uv\frac{\tan\varphi}{a}=-\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda}+fv+F_\lambda \ . \tag{3}$$

the hydrostatic equation

$$\frac{\partial\Phi}{\partial p}=-\alpha \ , \tag{4}$$

the first law of thermodynamic equation

$$\frac{dC_pT}{dt}-\alpha\omega=Q \ , \tag{5}$$

and the atmospheric state equation

$$p=\rho RT \ . \tag{6}$$

Reordering the gradient balance equation (1) yields

$$u^2=\left(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}-uf\right)\frac{a}{\tan\varphi} \quad . \tag{7}$$

To eliminate the derivatives of u with respect to t,φ and p in (3), differentiate (1) with respect to t,φ and p respectively, which generate the following formulas,

$$\frac{\partial u}{\partial t}=\frac{1}{f_A}\frac{\partial}{\partial t}\left(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}\right) , \tag{8}$$

$$\frac{\partial u}{\partial\varphi}=\frac{1}{f_A}\frac{\partial}{\partial\varphi}\left(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}\right)+\frac{1}{f_A}\left(\frac{2f}{\sin2\varphi}-a\beta\right)u-\frac{2}{f_A\sin2\varphi}\left(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}\right) , \tag{9}$$

$$\frac{\partial u}{\partial p}=\frac{1}{f_A}\frac{\partial}{\partial p}\left(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}\right) , \tag{10}$$

where

$$f_A=f+2u\frac{\tan\varphi}{a} \quad , \tag{11}$$

and then substitute (7)–(10) into (3). The result comes to be

$$\frac{1}{f_A}\left(\frac{\partial}{\partial t}+\frac{v}{a}\frac{\partial}{\partial\varphi}+\omega\frac{\partial}{\partial p}\right)\left(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}\right)+\frac{1}{f_A}\left(\frac{2vf}{a\sin2\varphi}-v\beta\right)u-\frac{2v}{f_A a\sin2\varphi}\left(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}\right)$$
$$+\frac{1}{2a\cos\varphi}\frac{a}{\tan\varphi}\frac{\partial}{\partial\lambda}\left(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}-uf\right)-uv\frac{\tan\varphi}{a}$$

$$= -\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda} + vf + F_\lambda \ . \tag{12}$$

Multiplying (12) by (11) gives

$$[(\frac{\partial}{\partial t} + \frac{v}{a}\frac{\partial}{\partial\varphi} + \omega\frac{\partial}{\partial p})(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}) + (\frac{2fv}{a\sin2\varphi} - v\beta)u - \frac{2v}{a\sin2\varphi}(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi})]$$
$$\text{(I)}$$
$$+ [\frac{1}{2a\cos\varphi}\frac{a}{\tan\varphi}(f + 2u\frac{\tan\varphi}{a})\frac{\partial}{\partial\lambda}(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi} - uf)] - [uv\frac{\tan\varphi}{a}(f + 2u\frac{\tan\varphi}{a})]$$
$$\text{(II)} \qquad\qquad\qquad \text{(III)}$$
$$= [(-\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda} + vf + F_\lambda)(f + 2u\frac{\tan\varphi}{a})] \ . \tag{13}$$
$$\text{(IV)}$$

To take the thermodynamic processes into account, rewriting the first law of thermodynamic equation (5)

$$\omega = (\frac{\partial\Phi}{\partial p})^{-1}(Q - \frac{dC_pT}{dt})$$
$$= (\frac{\partial\Phi}{\partial p})^{-1}(Q - \frac{\partial C_pT}{\partial t} - \frac{u}{a\cos\varphi}\frac{\partial C_pT}{\partial\lambda} - \frac{v}{a}\frac{\partial C_pT}{\partial\varphi} - \omega\frac{\partial C_pT}{\partial p}) \tag{14}$$

and then manipulating terms I, II, III and IV in (13) respectively with the use of (7) and (14) yield

$$\mathrm{I} = [\frac{2vf}{a\sin2\varphi} - v\beta - \frac{1}{a\cos\varphi}(\frac{\partial\Phi}{\partial p})^{-1}\frac{\partial C_pT}{\partial\lambda}\frac{\partial}{\partial p}(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi})]u$$
$$- [(\frac{\partial\Phi}{\partial p})^{-1}\frac{\partial C_pT}{\partial p}\frac{\partial}{\partial p}(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi})]\omega - \frac{2v}{a\sin2\varphi}(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi})$$
$$+ [(\frac{\partial}{\partial t} + \frac{v}{a}\frac{\partial}{\partial\varphi}) + (\frac{\partial\Phi}{\partial p})^{-1}(Q - \frac{\partial C_pT}{\partial t} - \frac{v}{a}\frac{\partial C_pT}{\partial\varphi})\frac{\partial}{\partial p}](-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}) \ ,$$
$$\mathrm{II} = \frac{f}{2\sin\varphi}\frac{\partial}{\partial\lambda}(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}) - \frac{f^2}{2\sin\varphi}\frac{\partial u}{\partial\lambda} + \frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}) - \frac{f}{2a\cos\varphi}\frac{\partial u^2}{\partial\lambda}$$
$$= \frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}) \ ,$$
$$\mathrm{III} = -fv\frac{\tan\varphi}{a}u - 2v\frac{\tan\varphi}{a}(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi} - uf) = fv\frac{\tan\varphi}{a}u - 2v\frac{\tan\varphi}{a}(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}) \ ,$$
$$\mathrm{IV} = f(-\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda} + fv + F_\lambda) + 2u\frac{\tan\varphi}{a}(-\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda} + fv + F_\lambda) \ .$$

According to the new forms of term I to term IV, (13) can be rewritten as

$$Du - E\omega = f(-\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda} + fv + F_\lambda) + \frac{2v}{a\sin2\varphi}(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}) - [\frac{\partial}{\partial t} + \frac{v}{a}\frac{\partial}{\partial\varphi}$$
$$+ (\frac{\partial\Phi}{\partial p})^{-1}(Q - \frac{\partial C_pT}{\partial t} - \frac{v}{a}\frac{\partial C_pT}{\partial\varphi})\frac{\partial}{\partial p} - \frac{2\tan\varphi}{a}v](-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi})] \ , \tag{15}$$

where the coefficients D and E are defined as

$$D = \frac{1}{a\cos\varphi}[\frac{\partial}{\partial\lambda} - (\frac{\partial\Phi}{\partial p})^{-1}\frac{\partial C_pT}{\partial\lambda}\frac{\partial}{\partial p}](-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi})$$

$$-\left[\frac{\tan\varphi}{a}\left(-\frac{2}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda}+fv+2F_\lambda\right)\right]+\frac{2vf}{a\sin2\varphi}-v\beta\ ,$$

$$E=\left(\frac{\partial\Phi}{\partial p}\right)^{-1}\frac{\partial C_pT}{\partial p}\frac{\partial}{\partial p}\left(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}\right)\ .$$

Coefficient D can be further simplified since those terms associated with f and β cancel each other out. Equation (15) is the linear diagnostic equation for the Walker circulation in terms of (u,ω). With given forcing on the right–hand side of the equation, (15) is not solvable yet because there are two unknowns u and ω. Therefore the two unknowns must be reduced to one. This is accomplished by introducing a stream function for the Walker circulation.

According to Stokes′ theorem, the Walker circulation is associated with the y–component of the curl of velocity. Vector analysis shows that the curl of velocity is non–divergent (Phillips, 1950), so the Walker circulation is only associated with the non–divergent part of velocity (u_ψ, ω_ψ) in the zonal plane. In spherical–isobaric coordinates, that is

$$\frac{1}{a\cos\varphi}\frac{\partial u_\psi}{\partial\lambda}+\frac{\partial\omega_\psi}{\partial p}=0\ . \tag{16}$$

It is easy to prove that formally the solutions of (16) are

$$u_\psi=\frac{\partial\psi}{\partial p},\quad \omega_\psi=-\frac{1}{a\cos\varphi}\frac{\partial\psi}{\partial\lambda}\ . \tag{17}$$

However, (16) is only part of the original continuity equation (2). To handle the rest of (2), a theorem is applied. The theorem states that if a vector function is piecewise differentiable everywhere and vanishes at infinity or outside a finite region, then it can be expressed as the sum of an ir–rotational vector and a non–divergent vector (Phillips, 1950; Weatherburn, 1966). Normally, a velocity field in the zonal plane is considered piecewise differentiable and vanishes at infinity, so this velocity can be represented as the sum of an ir–rotational vector and a non–divergent vector

$$\vec{v}=\vec{v}_\chi+\vec{v}_\psi,\quad \text{or}\quad u=u_\chi+v_\psi,\quad \omega=\omega_\chi+\omega_\psi\ . \tag{18}$$

With (16) and (18), the original continuity equation (2) can be rewritten as

$$\frac{1}{a\cos\varphi}\frac{\partial u_\chi}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial(v\cos\varphi)}{\partial\varphi}+\frac{\partial\omega_\chi}{\partial p}=0\ . \tag{19}$$

Up to now, (2) is partitioned into (16) and (19). Quantities u_χ and ω_χ in (19) are eliminated respectively from the candidates of horizontal and vertical branches of the Walker circulation. So, they must be predetermined and can only appear in (15) as given forcing factors (see Section 3).

Since the two unknowns for the Walker circulation now become u_ψ and ω_ψ, with (17), they can be replaced by the stream function ψ for the Walker circulation (one unknown). Substituting (17) and (18) into (15) yields

$$D\frac{\partial\psi}{\partial p}+E\frac{1}{a\cos\varphi}\frac{\partial\psi}{\partial\lambda}=f\left(-\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda}+fv+F_\lambda\right)+\frac{2v}{a\sin2\varphi}\left(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}\right)-Du_\chi+E\omega_\chi$$
$$-\left[\frac{\partial}{\partial t}+\frac{v}{a}\frac{\partial}{\partial\varphi}+\left(\frac{\partial\Phi}{\partial p}\right)^{-1}\left(Q-\frac{\partial C_pT}{\partial t}-\frac{v}{a}\frac{\partial C_pT}{\partial\varphi}\right)\frac{\partial}{\partial p}-\frac{2\tan\varphi}{a}v\right]\left(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}\right)\right]\ . \tag{20}$$

This is the final form of the linear diagnostic equation for the stream function ψ of the Walker circulation. With given forcing on the right of (20), the stream function ψ of the Walker circulation can be obtained through solving (20). Then the corresponding horizontal and vertical branches of the Walker circulation are calculated with the use of (17). Equation (20) reveals that the forcing associated with horizontal pressure gradient force, Coriolis force, frictional force, temperature advection, diabatic heating and the change of temperature with time will have impact on the Walker circulation. The effect of sea surface temperature on the Walker circulation will come from the lower boundary condition of temperature in this model domain. Since this diagnostic equation is a linear equation in terms of the stream function ψ, the numerical evaluation of the contribution from each individual forcing process will enable us to investigate the mechanisms responsible for the evolution of the Walker circulation.

The general analytic solution of the diagnostic equation (20) for the Walker circulation is (see the appendix)

$$\psi(x,p)=\psi_0[x(p_0)]+\int_{p_0}^{p} f\{g^*[g(x,p),\sigma],\sigma\}d\sigma \ ,$$

where $\psi_0[x(p_0)]$ stands for the given top (or bottom) vertical boundary value of the stream function and $f\{g^*[g(x,p),\sigma],\sigma\}$ is associated with the forcing function. This solution indicates that as long as the sign of the integrand varies, there will exist zonal circulations in a global zonal plane.

3. Predetermination of u_χ and ω_χ as forcing factors

Since u_χ and ω_χ are eliminated from the candidates of horizontal and vertical branches of the Walker circulation respectively, those terms associated with u_χ and ω_χ must be treated as forcing terms and predetermined in the diagnostic equation (20) for the stream function ψ of the Walker circulation. So far, there are two possible ways to determine u_χ and ω_χ. One results from the partitions of u and v into geostrophic and ageostrophic components

$$u=u_g+u_{ag}, \quad v=v_g+v_{ag} \ . \tag{21}$$

The assumption of constant Coriolis parameter gives

$$\frac{1}{a\cos\varphi}\frac{\partial u_g}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial(v_g\cos\varphi)}{\partial\varphi}=0 \ . \tag{22}$$

The substitution of (21) and (22) into (2) yields

$$\frac{1}{a\cos\varphi}\frac{\partial u_{ag}}{\partial\lambda}+\frac{1}{a\cos\varphi}\frac{\partial(v_{ag}\cos\varphi)}{\partial\varphi}+\frac{\partial\omega}{\partial p}=0 \ . \tag{23}$$

Equation (23) indicates that any vertical circulation (either the Walker circulation or the Hadley circulation) is mainly associated with ageostrophic wind. In other words, u_g, the geostrophic component of horizontal wind does not belong to the horizontal branch of the Walker circulation. Since u_χ is also non–indicator of the horizontal branch of the Walker circulation (see Section 2), there must exist a relationship between u_χ and u_g. One choice could be

$$u_\chi\approx u_g \ . \tag{24}$$

To determine the degree of this approximation, reexamine the continuity equation (19) in terms of (u_χ,v,ω_χ) and (23) in terms of (u_{ag},v_{ag},ω) with $f\approx$ constant. If $u_\chi\approx u_g$ is a good approximation, then the substitution of $u_\chi\approx u_g$ and its by–product $u_\psi\approx u_{ag}$ into (19) and (23) should lead to an identical form of continuity equation. That is exactly the case since on one hand, with the application of $u_\chi \approx u_g$ and (22), the continuity equation (19) becomes

$$\frac{1}{a\cos\varphi}\frac{\partial(v_{ag}\cos\varphi)}{\partial\varphi} + \frac{\partial\omega_\chi}{\partial p} = 0 \ . \tag{25}$$

On the other, with $u_\psi \approx u_{ag}$ and the application of (18) and (16), the continuity equation (23) becomes

$$\frac{1}{a\cos\varphi}\frac{\partial u_\psi}{\partial\lambda} + \frac{1}{a\cos\varphi}\frac{\partial(v_{ag}\cos\varphi)}{\partial\varphi} + \frac{\partial\omega_\psi}{\partial p} + \frac{\partial\omega_\chi}{\partial p}$$
$$= \frac{1}{a\cos\varphi}\frac{\partial(v_{ag}\cos\varphi)}{\partial\varphi} + \frac{\partial\omega_\chi}{\partial p} = 0 \ .$$

As what we expect, it ends up being the same as (25). Because the only assumption required in the above analysis is $f \approx$ constant, it is the only approximation involved in $u_\chi \approx u_g$. With (25), ω_χ can be obtained as

$$\omega_\chi\big|_p = -\int_{p_{top}}^{p}\frac{1}{a\cos\varphi}\left(\frac{\partial(v_{ag}\cos\varphi)}{\partial\varphi} - \frac{v_g\beta}{f}\right)dp \ , \tag{26}$$

with the effect of Coriolis parameter as a function of latitude and $\omega_\chi = 0$ at the top of model domain. Equations (24) and (26) present one of the possible ways to predetermine the forcing factors v_χ and ω_χ.

There is another way to determine u_χ and ω_χ. It starts with the calculation of "observed" y–component of vorticity $(\zeta_y)_{ob}$ with the use of "observed" zonal and vertical motions (u_{ob}, ω_{ob}). Then the Poisson equation $\nabla^2\psi_{ob} = (\zeta_y)_{ob}$ can be solved to yield the "observed" stream function ψ_{ob} associated with the Walker circulation (this ψ_{ob} field can serve as the standard solution to test the accuracy of the simulated steam function ψ field). With the use of (17) and this ψ_{ob} field, $(u_\psi, \omega_\psi)_{ob}$ can be computed, which are the corresponding "observed" zonal and vertical branches of the Walker circulation. Finally, the forcing factors u_χ and ω_χ are predetermined through $u_\chi = u_{ob} - (u_\psi)_{ob}$, and $\omega_\chi = \omega_{ob} - (\omega_\psi)_{ob}$.

The second approach is accurate theoretically. Numerically, however, the value of u_χ calculated in this way may be contaminated by the truncation errors, round–off errors and discretization errors to a higher extent (than the value of $u_\chi \approx u_g$) since a complex Poisson equation in the spherical–isobaric coordinates must be solved numerically in order to get u_χ. Besides, the vertical branch of the "observed" zonal circulation ω_{ob} would be calculated with many assumptions.

4. The derivation of a linear diagnostic equation for the meridionally–averaged Walker circulation

As for the meridionally averaged Walker circulation, a better gradient balance assumption is expected with the meridionally averaged properties u,φ and pressure gradient force, that is

$$\overline{u}\left(\overline{f} + \overline{u}\frac{\tan\overline{\varphi}}{a}\right) = -\frac{1}{a}\overline{\frac{\partial\Phi}{\partial\varphi}} \ , \tag{27}$$

where the meridional average and its deviation are defined as

$$\overline{()}=\frac{1}{\varphi_2-\varphi_1}\int_{\varphi_1}^{\varphi_2}()d\varphi,\quad ()'=()-\overline{()}\ . \tag{28}$$

If $\varphi_1=0^\circ$ and $\varphi_2=20^\circ$ are considered, then the averaged latitude will be $\overline{\varphi}=10^\circ$ with $0^\circ<|\varphi'|<10^\circ$. However, meridionally averaging the rest primitive equations in spherical–isobaric coordinates requires extra work since $\cos\varphi$ is involved in the denominator of several terms associated with unknown property u. To handle this problem, the Taylor expansion of $\cos\varphi\approx 1-\varphi^2/2!$ and the binomial theorem $(1+x)^{-1}\approx 1-x$ with $-1<x<1$ are used to get an approximate expression for $1/\cos\varphi$ under the condition that the value of $|\varphi'|$ is $\pi|\varphi'|/180<1$ in radian. The result is

$$\frac{1}{\cos\varphi}=\frac{1}{\cos(\overline{\varphi}+\varphi')}=\frac{1}{\cos\overline{\varphi}\cos\varphi'-\sin\overline{\varphi}\sin\varphi'}\approx\frac{1}{\cos\overline{\varphi}\cos\varphi'}$$
$$\approx\frac{1}{\cos\overline{\varphi}[1-\frac{(\varphi')^2}{2!}]}\approx\frac{1}{\cos\overline{\varphi}}[1+\frac{(\varphi')^2}{2!}]\ . \tag{29}$$

Obviously, the smaller the absolute values of φ and φ' are, the better this approximation will be. With (29), the meridionally–averaged continuity equation is approximately

$$\frac{1+0.5\overline{(\varphi')^2}}{a\cos\overline{\varphi}}\frac{\partial\overline{u}}{\partial\lambda}+\overline{\frac{1}{a\cos\varphi}\frac{\partial(v\cos\varphi)}{\partial\varphi}}+\frac{\partial\overline{\omega}}{\partial p}=0\ . \tag{30}$$

For the zonal motion equation, an approximate form for the meridionally–averaged zonal advection of momentum in (3) can be given by

$$\overline{\frac{1}{2a\cos\varphi}\frac{\partial u^2}{\partial\lambda}}\approx\overline{\frac{1}{2a\cos\overline{\varphi}}(1+\frac{(\varphi')^2}{2!})\frac{\partial(\overline{u}+u')^2}{\partial\lambda}}$$
$$\approx\frac{1}{2a\cos\overline{\varphi}}[\frac{\partial(\overline{u})^2}{\partial\lambda}+\overline{\frac{\partial(u')^2}{\partial\lambda}}]+\frac{1}{2a\cos\overline{\varphi}}\frac{\overline{(\varphi')^2}}{2!}\frac{\partial(\overline{u})^2}{\partial\lambda}$$
$$\approx\frac{1+0.5\overline{(\varphi')^2}}{2a\cos\overline{\varphi}}\frac{\partial(\overline{u})^2}{\partial\lambda}+\frac{1}{2a\cos\overline{\varphi}}\frac{\partial\overline{(u')^2}}{\partial\lambda}\ .$$

Applying the binomial theorem to the fifth term on the left–hand of (3) with $0<\tan\overline{\varphi}\tan|\varphi'|<1$, this term becomes

$$\overline{uv\frac{\tan\varphi}{a}}=\overline{(\overline{u}\overline{v}+\overline{u}v'+\overline{v}u'+u'v')\frac{\tan\overline{\varphi}+\tan\varphi'}{a(1-\tan\overline{\varphi}\tan\varphi')}}$$
$$\approx\overline{(\overline{u}\overline{v}+\overline{u}v'+\overline{v}u'+u'v')(1+\tan\overline{\varphi}\tan\varphi')\frac{\tan\overline{\varphi}+\tan\varphi'}{a}}$$
$$\approx\overline{u}\{\overline{v}\frac{\tan\overline{\varphi}}{a}[1+\overline{(\tan\varphi')^2}]+\frac{\overline{v'\tan\varphi'}}{a}[1+(\tan\overline{\varphi})^2]\}$$
$$+\overline{v}\frac{\overline{u'\tan\varphi'}}{a}[1+(\tan\overline{\varphi})^2]+\overline{u'v'}\frac{\tan\overline{\varphi}}{a}\ .$$

Adopting the above approximate forms, the meridionally–averaged zonal motion equation becomes

$$\frac{\partial\overline{u}}{\partial t}+\overline{\omega}\frac{\partial\overline{u}}{\partial p}+\frac{1+0.5\overline{(\varphi')^2}}{2a\cos\overline{\varphi}}\frac{\partial\overline{u}^2}{\partial\lambda}-\overline{u}\{\overline{v}\frac{\tan\overline{\varphi}}{a}[1+\overline{(\tan\varphi')^2}]$$

$$+ \overline{\frac{v'\tan\varphi'}{a}}[1+(\tan\overline{\varphi})^2]\} + \frac{1}{2a\cos\overline{\varphi}}\frac{\partial\overline{u'^2}}{\partial\lambda} + \overline{\frac{v'}{a}\frac{\partial u'}{\partial\varphi}}$$

$$+ \overline{\omega'\frac{\partial u'}{\partial p}} - \overline{v}\overline{\frac{u'\tan\varphi'}{a}}[1+(\tan\overline{\varphi})^2] - \overline{u'v'}\frac{\tan\overline{\varphi}}{a}$$

$$= -\overline{\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda}} + \overline{f}\overline{v} + \overline{F}_\lambda \ . \tag{31}$$

In the same way, the meridionally averaged zonal advection of temperature in the first law of thermodynamic equation (14) can be approximately expressed as

$$\overline{\frac{u}{a\cos\varphi}\frac{\partial C_pT}{\partial\lambda}} \approx \overline{\frac{(\overline{u}+u')}{a\cos\overline{\varphi}}(1+\frac{(\varphi')^2}{2!})\frac{\partial C_p(\overline{T}+T')}{\partial\lambda}}$$

$$\approx \frac{[1+0.5\overline{(\varphi')^2}]\overline{u}}{a\cos\overline{\varphi}}\frac{\partial C_p\overline{T}}{\partial\lambda} + \overline{\frac{u'}{a\cos\overline{\varphi}}\frac{\partial C_pT'}{\partial\lambda}} \ .$$

So the first law of thermodynamic equation comes to be

$$\overline{\omega} = (\overline{\frac{\partial\Phi}{\partial p}})^{-1}\{\overline{Q} - \frac{\partial C_p\overline{T}}{\partial t} - \frac{[1+0.5\overline{(\varphi')^2}]\overline{u}}{a\cos\overline{\varphi}}\frac{\partial C_p\overline{T}}{\partial\lambda} - \overline{\omega}\frac{\partial C_p\overline{T}}{\partial p}$$

$$- \overline{\frac{u'}{a\cos\overline{\varphi}}\frac{\partial C_pT'}{\partial\lambda}} - \overline{\frac{v'}{a}\frac{\partial C_pT'}{\partial\varphi}} - \overline{\omega'\frac{\partial C_pT'}{\partial p}} - \overline{\frac{\partial\Phi'}{\partial p}\omega'} \ . \tag{32}$$

Reordering the gradient balance equation (27) yields

$$\overline{u}^2 = (-\frac{1}{a}\overline{\frac{\partial\Phi}{\partial\varphi}} - \overline{u}\overline{f})\frac{a}{\tan\overline{\varphi}} \ . \tag{33}$$

Differentiate (27) with respect to t and p respectively yields

$$\frac{\partial\overline{u}}{\partial t} = \frac{1}{\overline{f_A}}\frac{\partial}{\partial t}(-\frac{1}{a}\overline{\frac{\partial\Phi}{\partial\varphi}}) \ , \tag{34}$$

$$\frac{\partial\overline{u}}{\partial p} = \frac{1}{\overline{f_A}}\frac{\partial}{\partial p}(-\frac{1}{a}\overline{\frac{\partial\Phi}{\partial\varphi}}) \ , \tag{35}$$

where

$$\overline{f_A} = \overline{f} + 2\overline{u}\frac{\tan\overline{\varphi}}{a} \ . \tag{36}$$

The substitution of (33)–(35) into (31) gives

$$\frac{1}{\overline{f_A}}(\frac{\partial}{\partial t} + \overline{\omega}\frac{\partial}{\partial p})(-\frac{1}{a}\overline{\frac{\partial\Phi}{\partial\varphi}}) + \frac{1+0.5\overline{(\varphi')^2}}{2\sin\varphi}\frac{\partial}{\partial\lambda}(-\frac{1}{a}\frac{\partial\Phi}{\partial\varphi} - \overline{u}\overline{f})$$

$$\overline{u}\{\overline{v}\frac{\tan\overline{\varphi}}{a}[1+\overline{(\tan\varphi')^2}] + \overline{\frac{v'\tan\varphi'}{a}}[1+(\tan\overline{\varphi})^2]\}$$

$$+ \frac{1}{2a\cos\overline{\varphi}}\overline{\frac{\partial(u')^2}{\partial\lambda}} + \overline{\frac{v'}{a}\frac{\partial u'}{\partial\varphi}} + \overline{\omega'\frac{\partial u'}{\partial p}} - \overline{v}\overline{\frac{u'\tan\varphi'}{a}}[1+(\tan\overline{\varphi})^2] - \overline{u'v'}\frac{\tan\overline{\varphi}}{a}$$

$$= -\overline{\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda}} + \overline{v}\overline{f} + \overline{F}_\lambda \ . \tag{37}$$

Multiplying (37) by (36) yields

$$\overline{\omega}\frac{\partial}{\partial p}(-\frac{1}{a}\frac{\partial\overline{\Phi}}{\partial\varphi})+\frac{\partial}{\partial t}(-\frac{1}{a}\frac{\partial\overline{\Phi}}{\partial\varphi})+[\frac{1+0.5\overline{(\varphi')^2}}{2\sin\overline{\varphi}}\frac{\partial}{\partial\lambda}(-\frac{1}{a}\frac{\partial\overline{\Phi}}{\partial\varphi}-\overline{u}\overline{f})(\overline{f}+2\overline{u}\frac{\tan\overline{\varphi}}{a})]$$
$$\text{(I)} \qquad \text{(II)} \qquad \text{(III)}$$
$$-\overline{u}\{\overline{v}\frac{\tan\overline{\varphi}}{a}[1+\overline{(\tan\varphi')^2}]+\frac{\overline{v'\tan\varphi'}}{a}[1+(\tan\overline{\varphi})^2]\}(\overline{f}+2\overline{u}\frac{\tan\overline{\varphi}}{a})$$
$$\text{(IV)}$$
$$+\{\frac{1}{2a\cos\overline{\varphi}}\frac{\partial\overline{(u')^2}}{\partial\lambda}+\overline{\frac{v'}{a}\frac{\partial u'}{\partial\varphi}}+\overline{\omega'\frac{\partial u'}{\partial p}}-\overline{v}\frac{\overline{v'\tan\varphi'}}{a}[1+(\tan\overline{\varphi})^2]-\overline{u'v'}\frac{\tan\overline{\varphi}}{a}\}(\overline{f}+2\overline{u}\frac{\tan\overline{\varphi}}{a})$$
$$\text{(V)}$$
$$=(-\overline{\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda}}+\overline{v}\overline{f}+\overline{F}_\lambda)(\overline{f}+2\overline{u}\frac{\tan\overline{\varphi}}{a}) \; . \tag{38}$$
$$\text{(VI)}$$

With the use of (32), term (I) on the left side of (38) becomes

$$\mathrm{I}=\left[\begin{array}{c}(\frac{\partial\overline{\Phi}}{\partial p})^{-1}\{\overline{Q}-\frac{\partial C_p\overline{T}}{\partial t}-\frac{[1+0.5\overline{(\varphi')^2}]\overline{u}}{a\cos\overline{\varphi}}\frac{\partial C_p\overline{T}}{\partial\lambda}-\overline{\omega}\frac{\partial C_p\overline{T}}{\partial p}\\ -\overline{\frac{u'}{a\cos\overline{\varphi}}\frac{\partial C_pT'}{\partial\lambda}}-\overline{\frac{v'}{a}\frac{\partial C_pT'}{\partial\varphi}}-\overline{\omega'\frac{\partial C_pT'}{\partial p}}-\overline{\frac{\partial\Phi'}{\partial p}\omega'}\end{array}\right]\frac{\partial}{\partial p}(-\frac{1}{a}\frac{\partial\overline{\Phi}}{\partial\varphi}) \; .$$

With the use of (33), terms (III) and (IV) on the left side of (38) become respectively

$$\mathrm{III}=\frac{[1+0.5\overline{(\varphi')^2}]\overline{u}}{a\cos\overline{\varphi}}\frac{\partial}{\partial\lambda}(-\frac{1}{a}\frac{\partial\overline{\Phi}}{\partial\varphi}) \; ,$$
$$\mathrm{IV}=\overline{f}\{\overline{v}\frac{\tan\overline{\varphi}}{a}[1+\overline{(\tan\varphi')^2}]+\frac{\overline{v'\tan\varphi'}}{a}[1+(\tan\overline{\varphi})^2]\}\overline{u}$$
$$-2\{\overline{v}\frac{\tan\overline{\varphi}}{a}[1+\overline{(\tan\varphi')^2}]+\frac{\overline{v'\tan\varphi'}}{a}[1+(\tan\overline{\varphi})^2]\}(-\frac{1}{a}\frac{\partial\overline{\Phi}}{\partial\varphi}) \; .$$

The substitution of these three terms into (38) yields

$$\overline{D}\overline{u}-\overline{E}\overline{\omega}$$
$$=\overline{f}(-\overline{\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda}}+\overline{f}_v+\overline{F}_\lambda)-\overline{f}\{-\frac{1}{2a\cos\overline{\varphi}}\frac{\partial\overline{(u')^2}}{\partial\lambda}-\overline{\frac{v'}{a}\frac{\partial u'}{\partial\varphi}}-\overline{\omega'\frac{\partial u'}{\partial p}}$$
$$-\overline{v}\frac{\overline{v'\tan\varphi'}}{a}[1+(\tan\overline{\varphi})^2]-\overline{u'v'}\frac{\tan\overline{\varphi}}{a}\}-\{\frac{\partial}{\partial t}+(\frac{\partial\overline{\Phi}}{\partial p})^{-1}[\overline{Q}-\frac{\partial C_p\overline{T}}{\partial t}$$
$$-\overline{\frac{u'}{a\cos\overline{\varphi}}\frac{\partial C_pT'}{\partial\lambda}}-\overline{\frac{v'}{a}\frac{\partial C_pT'}{\partial\varphi}}-\overline{\omega'\frac{\partial C_pT'}{\partial p}}-\overline{\omega'\frac{\partial\Phi'}{\partial\varphi}}]\frac{\partial}{\partial p}$$
$$-\frac{2\overline{v}\tan\overline{\varphi}}{a}[1+\overline{(\tan\varphi')^2}]-\frac{2\overline{v'\tan\varphi'}}{a}[1+(\tan\overline{\varphi})^2]\}(-\frac{1}{a}\frac{\partial\overline{\Phi}}{\partial\varphi}) \; , \tag{39}$$

where the coefficients D and E are defined as

$$\overline{D}=\frac{1+0.5\overline{(\varphi')^2}}{a\cos\overline{\varphi}}[\frac{\partial}{\partial\lambda}-(\frac{\partial\overline{\Phi}}{\partial\lambda})^{-1}\frac{\partial C_p\overline{T}}{\partial\lambda}\frac{\partial}{\partial p}](-\frac{1}{a}\frac{\partial\overline{\Phi}}{\partial\varphi})$$
$$-\frac{2\tan\overline{\varphi}}{a}[-\overline{\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda}}+\overline{f}_v+\overline{F}_\lambda-\frac{1}{2a\cos\overline{\varphi}}\frac{\partial\overline{(u')^2}}{\partial\lambda}-\overline{\frac{v'}{a}\frac{\partial u'}{\partial\lambda}}-\overline{\omega'\frac{\partial u'}{\partial p}}-\overline{u'v'}\frac{\tan\overline{\varphi}}{a}]$$
$$-\overline{v}\frac{\overline{u'\tan\varphi'}}{a}[1+(\tan\overline{\varphi})^2]+\overline{f}\{\overline{v}\frac{\tan\overline{\varphi}}{a}[1+\overline{(\tan\varphi')^2}]+\frac{\overline{v'\tan\varphi'}}{a}[1+(\tan\overline{\varphi})^2]\} \; ,$$

$$\overline{E}=(\overline{\frac{\partial\Phi}{\partial p}})^{-1}\frac{\partial C_p\overline{T}}{\partial p}\frac{\partial}{\partial p}(-\frac{1}{a}\frac{\overline{\partial\Phi}}{\partial\varphi}) .$$

With

$$\overline{u}=\overline{u}_{\psi}+\overline{u}_{\chi},\quad \overline{\omega}=\overline{\omega}_{\psi}+\overline{\omega}_{\chi} \tag{40}$$

the continuity equation (30) is approximately split into two parts

$$\frac{1+0.5\overline{(\varphi')^2}}{a\cos\overline{\varphi}}\frac{\partial\overline{u}_{\chi}}{\partial\lambda}+\frac{\partial\overline{\omega}_{\chi}}{\partial p}=-\overline{\frac{1}{a\cos\varphi}\frac{\partial(v\cos\varphi)}{\partial\varphi}} , \tag{41}$$

$$\frac{1+0.5\overline{(\varphi')^2}}{a\cos\overline{\varphi}}\frac{\partial\overline{u}_{\psi}}{\partial\lambda}+\frac{\partial\overline{\omega}_{\psi}}{\partial p}=0 . \tag{42}$$

It is easy to show that the formal solutions of (42) are

$$\overline{u}_{\psi}=\frac{\partial\overline{\Psi}}{\partial p} , \tag{43}$$

$$\overline{\omega}_{\psi}=-\frac{1+0.5\overline{(\varphi')^2}}{a\cos\overline{\varphi}}\frac{\partial\overline{\Psi}}{\partial\lambda} . \tag{44}$$

After substituting (43) and (44) into (39), the linear diagnostic equation for the meridionally–averaged stream function of Walker circulation emerges as

$$\begin{aligned}
&\overline{D}\frac{\partial\overline{\Psi}}{\partial p}+\frac{[1+0.5\overline{(\varphi')^2}]\overline{E}}{a\cos\overline{\varphi}}\frac{\partial\overline{\Psi}}{\partial\lambda}\\
&=\overline{f}(-\overline{\frac{1}{a\cos\varphi}\frac{\partial\Phi}{\partial\lambda}}+\overline{f}_v+\overline{F}_{\lambda})-\overline{f}\{-\overline{\frac{1}{2a\cos\overline{\varphi}}\frac{\partial(u')^2}{\partial\lambda}}-\overline{\frac{v'}{a}\frac{\partial u'}{\partial\varphi}}-\overline{\omega'\frac{\partial u'}{\partial p}}-\overline{u'v'}\frac{\tan\overline{\varphi}}{a}\\
&\quad-\overline{v}\frac{\overline{u'\tan\varphi'}}{a}[1+(\tan\overline{\varphi})^2]\}-\overline{D}\overline{u}_{\chi}+\overline{E}\overline{\omega}_{\chi}\\
&\quad-\{\frac{\partial}{\partial t}+(\overline{\frac{\partial\Phi}{\partial p}})^{-1}[\overline{Q}-\frac{\partial C_p\overline{T}}{\partial t}-\overline{\frac{u'}{a\cos\overline{\varphi}}\frac{\partial C_pT'}{\partial\lambda}}-\overline{\frac{v'}{a}\frac{\partial C_pT'}{\partial\varphi}}-\overline{\omega'\frac{\partial C_pT'}{\partial p}}-\overline{\omega'\frac{\partial\Phi'}{\partial\varphi}}]\frac{\partial}{\partial p}\\
&\quad-\frac{2\overline{v}\tan\overline{\varphi}}{a}[1+\overline{(\tan\varphi')^2}]-\frac{2\overline{v'\tan\varphi'}}{a}[1+(\tan\overline{\varphi})^2]\}(-\frac{1}{a}\frac{\overline{\partial\Phi}}{\partial\varphi}) ,
\end{aligned} \tag{45}$$

where the given forcing factors: the averaged u_{χ} and ω_{χ} are approximately

$$\begin{aligned}
&\overline{u}_{\chi}=\overline{u}_{g} ,\\
&\overline{\omega}_{\chi}\big|_p=-\int_{p_{\rm top}}^{p}[\frac{1+0.5\overline{(\varphi')^2}}{a\cos\overline{\varphi}}\frac{\partial\overline{u}_g}{\partial\lambda}+\overline{\frac{1}{a\cos\varphi}\frac{\partial v\cos\varphi}{\partial\varphi}}]dp .
\end{aligned} \tag{46}$$

5. Remark

As part of general circulation, the Walker circulation interacts with other important dynamic and thermodynamic atmospheric systems to modify the global weather. To take these dynamic and thermodynamic processes into account, the derivation of diagnostic equations for the Walker circulation starts with all atmospheric equations in their primitive forms except for the gradient balance form for the meridional motion equation.

The bias due to the gradient balance assumption may be reduced with meridionally–averaged quantities. However, with this meridional average approach, not only extra work is needed but the average forms of equations are merely approximate for those primitive equations including the product of zonal motion u and $1/(\cos\varphi)$. For this reason, no comment is made on which version is better: the diagnostic equation for the Walker circulation along the individual latitude or the diagnostic equation for the meridionally–averaged Walker circulation over a tropical zone before any quantitative result is reached.

The next step of this study will be designing finite difference equations for numerically solving those two diagnostic equations and followed by some case studies.

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Appendix

The General Analytic Solution of the Diagnostic Equation for the Walker Circulation

The diagnostic equation for the Walker circulation (20) or (45) can be generalized as

$$\frac{\partial\psi}{\partial p} + a(x,p)\frac{\partial\psi}{\partial x} = f(x,p) \ , \tag{A1}$$

$$\psi(x,p_0) = \psi_0(x) \tag{A2}$$

with periodic boundary in x coordinate, where ψ_0 stands for the top (or bottom) boundary value of ψ. To solve (A1) and (A2), the variables (x,p) are replaced by (ξ,τ) with the relations

$$\tau = p \ , \tag{A3}$$

$$\xi = g(x,p) \ , \tag{A4}$$

where, according to strikwerda (1989), $\xi = g(x,p)$ is determined by

$$\frac{dx}{d\tau} = a(x,p) \qquad \text{with} \qquad x(\tau_0) = \xi \ . \tag{A5}$$

Correspondingly, there exists a relationship between ψ and ψ^* :

$$\psi(x,p) = \psi^*(\xi,\tau) \ ,$$

where ψ and ψ^* represent the same stream function but in different coordinates. The purpose of changing the variables from (x,p) to (ξ,τ) in this way is to obtain the characteristic curves $x(\tau)= g^*(\xi,\tau)$ or $\xi(p)= g(x,p)$ along which the stream function $\psi^*(\xi,\tau)$ satisfies an 'ordinary' differential equation (A6) instead of the partial differential equation like (A1) since with (A3)–(A5), the derivative of ψ^* with respect to τ is

$$\frac{\partial\psi^*}{\partial\tau}=\frac{\partial\psi}{\partial p}\frac{dp}{d\tau}+\frac{\partial\psi}{\partial x}\frac{dx}{d\tau}=\frac{\partial\psi}{\partial p}+a(x,p)\frac{\partial\psi}{\partial x}=f(x,p)\ , \tag{A6}$$

$$\frac{\partial\psi^*}{\partial\tau}=f\,[g^*(\xi,\tau),\tau]\ .$$

The solution of (A6) is

$$\psi^*(\xi,\tau)=\psi^*(\xi,\tau_0)+\int_{\tau_0}^{\tau} f[g^*(\xi,\sigma),\sigma]d\sigma\ . \tag{A7}$$

With (A3), (A4), (A5) and (A7), the solution of (A1) and (A2) is

$$\begin{aligned}\psi(x,p)&=\psi_0[g(x,p)]+\int_{p_0}^{p} f\{[g^*[g(x,p),\sigma],\sigma\}d\sigma\\ &=\psi_0[x(p_0)]+\int_{p_0}^{p} f\{[g^*[g(x,p),\sigma],\sigma\}d\sigma\end{aligned} \tag{A8}$$

with periodic boundary in x coordinate.

Example: Find the solution for the equations with variable coefficient

$$\frac{\partial\psi}{\partial p}+x\frac{\partial\psi}{\partial x}=x\ , \tag{A9}$$

$$\psi(x,0)=\psi_0(x)\ . \tag{A10}$$

Following the above procedure, the corresponding characteristic curve can be determined exactly by solving

$$\frac{dx}{d\tau}=x \qquad \text{with} \qquad x(0)=\xi\ . \tag{A11}$$

The result turns out to be

$$x(\tau)=\xi e^{\tau}=g^*(\xi,\tau) \qquad \text{or} \qquad \xi=xe^{-p}=g(x,p)\ . \tag{A12}$$

With (A12), the differential equation for ψ^* becomes

$$\frac{\partial\psi^*}{\partial\tau}=x=\xi e^{\tau}\ . \tag{A13}$$

Therefore, the solution of (A9) and (A10) is

$$\psi(x,p)=\psi^*(\xi,\tau)=\psi_0(\xi)+\int_0^p \xi e^{\sigma}d\sigma\ ,$$

$$\begin{aligned}\psi(x,p)&=\psi_0(xe^{-p})+xe^{-p}(e^{p}-1)\\ &=\psi_0(xe^{-p})+x(1-e^{-p})\ .\end{aligned} \tag{A14}$$

It can be verified that (A14) satisfies (A9) and (A10).

Prove: According to (A14), the derivatives of ψ with respect to x and p respectively become

$$\frac{\partial\psi}{\partial p}=\frac{\partial\psi_0}{\partial g}\frac{\partial g}{\partial p}+xe^{-p}=-xe^{-p}\frac{\partial\psi_0}{\partial g}+xe^{-p}\ , \tag{A15}$$

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi_0}{\partial g}\frac{\partial g}{\partial x}+1-e^{-p}=e^{-p}\frac{\partial\psi_0}{\partial g}+1-e^{-p}\ . \tag{A16}$$

The substitution of (A15) and (A16) into the left side of (A9) yields

$$\frac{\partial\psi}{\partial p}+x\frac{\partial\psi}{\partial x}=(-x+x)e^{-p}\frac{\partial\psi_0}{\partial g}+xe^{-p}-xe^{-p}+x=x\ . \tag{A17}$$

With $p=0$, (A14) gives

$$\psi(x,0)=\psi_0(x)\ .$$

Walker 环流诊断方程

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摘　要

导出了两个 Walker 环流线性诊断方程，一个适用于数值模拟热带地区某一纬度的 Walker 环流；另一个适用于数值模拟经向平均的 Walker 环流。推导所用的基本方程组中，除了经向运动方程用梯度风平衡方程取代外，其余方程都是球—p 坐标系的原始方程。

关键词: Walker 环流，ENSO