Nonlinear Saturation of Baroclinic Instability in the Phillips Model: The Case of Energy

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ABSTRACT

A conservation law for the Phillips model is derived. Using this law, the nonlinear saturation of purely baroclinic instability caused by the vertical velocity shear of the basic flow in the Phillips model—the case of energy—is studied within the context of Arnold's second stability theorem. Analytic upper bounds on the energy of wavy disturbances are obtained. For one unstable region in the parameter plane, the result here is a second-order correction in ε to Shepherd's; For another unstable region, the analytic upper bound on the energy of wavy disturbances offers an effective constraint on wavy (nonzonal) disturbances Φ'_i at any time.

Key words: nonlinear saturation, baroclinic instability, Phillips model

1. Introduction

Ever since Shepherd suggested a novel method to study nonlinear saturation of barotropic and baroclinic instabiliy (Shepherd 1988a, 1993), many authors have made further research on this subject, for example, Zhu and Strobel (1992). In the meantime, Zeng (1989) studied the saturation of linear and nonlinear Haurwitz waves in a different way. However, most of the above works have applied only Arnold's first theorem. In recent years, Mu (1991), Mu and Shepherd (1994a, 1994b), and Mu et al. (1994) have established nonlinear stability criteria corresponding to Arnold's second theorem for some important models in atmospheric and oceanic dynamics. So, Arnold's second theorem as well as Arnold's first theorem can be taken into account in studying nonlinear saturation of instability. Paret and Vanneste (1996), for instance, using both Arnold's first and second theorems, have studied the nonlinear saturation of baroclinic instability in the three-layer Phillips model numerically. Xiang and Mu (1997) have studied the nonlinear saturation of instability of parallel shear flow analytically in the same way. Shepherd (1988a, 1993), within the context of Arnold's first theorem, investigated the nonlinear saturation of baroclinic instability in the two-layer models and obtained upper bounds on the potential enstrophy and energy of wavy disturbances, which in particular includes the results of the Phillips model. It is better to study the nonlinear saturation of baroclinic instability in the Phillips model within the context of Arnold's second theorem, because for the Phillips model, Arnold's second theorem covers the first theorem, Xiang and Lin

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(henceforth referred to as XL), using Arnold's second theorem, study the nonlinear saturation of purely baroclinic instability caused by the vertical velocity shear of the basic flow in the Phillips model, and obtain an upper bound on the potential enstrophy of wavy (nonzonal) disturbances. So, the present paper, following XL, within the context of Arnold's second theorem, studies the nonlinear saturation of purely baroclinic instability caused by the vertical velocity shear of the basic flow in the Phillips model: the case of energy. Recurring to a conservation law, an analytic upper bound on the energy of wavy disturbances is obtained.

The plan of this paper is as follows. In section 2, a brief review of the nonlinear stability criterion in the Phillips model is presented and a conservation law is derived. The upper bound on the energy of wavy disturbances to an initial zonal mean unstable flow is obtained in section 3. Comparisons between the results in this paper and those in other papers are made in section 4. And finally, conclusions and discussion are given in section 5.

2. Nonlinear stability criterion in the Phillips model and one related conservation law

In this section, part of the results in Mu et al. (1994), for convenience in the forthcoming discussion, are briefly reviewed and concretized; readers may refer to Mu et al. (1994) for more details.

Mu et al. (1994) studied nonlinear stability of multilayer quasigeostrophic flows in the context of Arnold's second theorem, and obtained a nonlinear stability criterion. In particular, for the Phillips model, with the basic flow ($F_1 = F_2 = F$)

$$\Psi_i(y) = -U_i y,\ Q_i(y) = [(-1)^{i+1} F U_s + \beta]y, \qquad i = 1, 2, \tag{2.1}$$

where Ψ_1, Ψ_2 are the stream functions in the upper and lower layers respectively, Q_1, Q_2 are the corresponding potential vorticities, F is the rotational Froude number, and $U_s = U_1 - U_2$ (U_i are constants), the criterion reads specifically: any one of the following three conditions is a sufficient condition for nonlinear stability of the basic flow.

$$\text{(a)} \quad U_s^2 = \frac{\beta^2}{F^2} \quad \text{and} \quad \lambda^2 > 2F^2 , \tag{2.2}$$

$$\text{(b)} \quad U_s^2 < \frac{\beta^2}{F^2} , \tag{2.3}$$

$$\text{(c)} \quad U_s^2 > \frac{\beta^2}{F^2} , \quad \text{and}$$

$$\text{(c1)}\ \lambda^2 > 4F^2 \quad \text{or} \quad \text{(c2)}\ \ 2F^2 < \lambda^2 < 4F^2 \quad \text{and} \quad U_s^2 < \frac{4F^2\beta^2}{\lambda^2(4F^2 - \lambda^2)} . \tag{2.4}$$

Set

$$U_c^2 = \frac{4F^4}{\lambda^2(4F^2 - \lambda^2)} = \frac{4}{\left(\frac{\lambda}{F}\right)^2\left[4 - \left(\frac{\lambda}{F}\right)^2\right]} . \tag{2.5}$$

Then, the stability feature of the Phillips model is displayed in Fig. 1. Mu (1998) further pointed out that the above criterion is optimal in the following sense. If it is violated, except in several critical cases, there is at least a certain zonal periodic channel where there exist zonal normal modes growing exponentially with time.

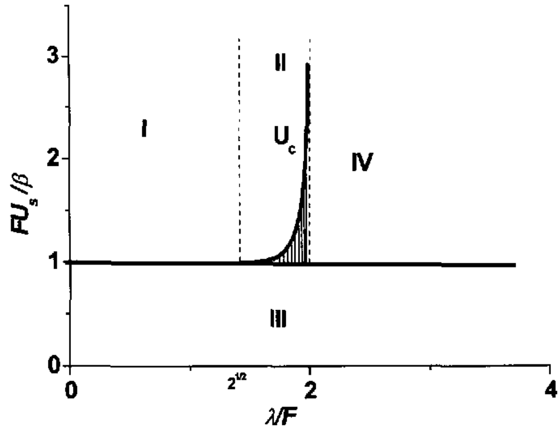

Fig. 1. The features of nonlinear stability in the Phillips model. Regions I and II are possibly nonlinearly unstable; region III is nonlinearly stable, not only by Arnold's first theorem, but also by Arnold's second theorem; region IV is nonlinearly stable by Arnold's second theorem only.

The above criterion comes from the following conservation law

$$\frac{d}{dt}[E(t)+A(t)]=0\ , \tag{2.6}$$

where

$$E(t)=\int_\Omega \frac{d}{2}[|\nabla\psi_1|^2+|\nabla\psi_2|^2+F(\psi_2-\psi_1)^2]dxdy \tag{2.7}$$

is the disturbance energy, while

$$A(t)=\int_\Omega \{d\sum_{i=1}^{2}[G_i(Q_i+q_i)-G_i(Q_i)-G'_i(Q_i)q_i]\}dxdy\ . \tag{2.8}$$

The terms q_i in (2.8) are the disturbance potential vorticities.
For the Phillips model, when U_1, U_2 satisfy

$$U_s^2<\frac{\beta^2}{F^2}\ , \tag{2.9}$$

i.e.,

$$\beta-FU_s>0,\quad \beta+FU_s>0\ , \tag{2.10}$$

the functions $G_i(\eta)$ and $\Psi_i^\alpha(\eta)$ are defined according to

$$G_i(\eta)=\int^\eta \Psi_i^\alpha(\tau)d\tau\ , \tag{2.11}$$

$$\Psi_1+\alpha y=\frac{\alpha-U_1}{\beta+FU_s}Q_1\equiv\Psi_1^\alpha(Q_1)\ , \tag{2.12}$$

$$\Psi_2+\alpha y=\frac{\alpha-U_2}{\beta-FU_s}Q_2\equiv\Psi_2^\alpha(Q_2)\ , \tag{2.13}$$

where α is a parameter. Then, (2.6) becomes

$$E(t)+\int_{\Omega}\frac{d}{2}\left(\frac{\alpha-U_1}{\beta+FU_s}q_1^2+\frac{\alpha-U_2}{\beta-FU_s}q_2^2\right)dxdy=$$

$$E(0)+\int_{\Omega}\frac{d}{2}\left(\frac{\alpha-U_1}{\beta+FU_s}q_{10}^2+\frac{\alpha-U_2}{\beta-FU_s}q_{20}^2\right)dxdy\ . \tag{2.14}$$

Because of (2.9), α may be any sufficiently negative number (Mu et al. 1994, section 4.2 case 3). Divide the two sides of (2.14) by α, and let $\alpha\to-\infty$. Then (2.14) becomes

$$\int_{\Omega}\left(\frac{q_1^2}{\beta+FU_s}+\frac{q_2^2}{\beta-FU_s}\right)dxdy=\int_{\Omega}\left(\frac{q_{10}^2}{\beta+FU_s}+\frac{q_{20}^2}{\beta-FU_s}\right)dxdy\ . \tag{2.15}$$

Noting that

$$\int_{\Omega}q_i^2\,dxdy=\int_{\Omega}q_i^{*2}\,dxdy+\int_{\Omega}q_i'^2\,dxdy\ , \tag{2.16}$$

where

$$q_i^*=\frac{\int_{\Omega}q_{i0}\,dxdy}{\int_{\Omega}dxdy}\ ,\quad q_i'=q_i-q_i^*\ ,$$

one has

$$\int_{\Omega}\left(\frac{q_1'^2}{\beta+FU_s}+\frac{q_2'^2}{\beta-FU_s}\right)dxdy=\int_{\Omega}\left(\frac{q_{10}'^2}{\beta+FU_s}+\frac{q_{20}'^2}{\beta-FU_s}\right)dxdy\ . \tag{2.17}$$

Equation (2.17) is a conservation law for disturbances. Substituting (2.16) into (2.14), and using (2.17) yields

$$E(t)+\frac{U_s}{\beta-FU_s}\int_{\Omega}\frac{d}{2}q_2'^2\,dxdy=E(0)+\frac{U_s}{\beta-FU_s}\int_{\Omega}\frac{d}{2}q_{20}'^2\,dxdy\ . \tag{2.18}$$

Equation (2.18) will play a key role below.

3. Upper bounds on the energy of wavy disturbances to an unstable basic flow

In this section, the conservation law (2.18) is used to study the nonlinear saturation of purely baroclinic instability in the Phillips model, i.e., to obtain upper bounds on the energy of wavy disturbances to an unstable basic flow.

Consider the total flow (Φ_i, P_i), $i=1,2$, with the initial condition

$$\Phi_{i0}=\overline{\Phi}_{i0}+\Phi'_{i0}\ , \tag{3.1}$$

$$\overline{\Phi}_{10}=-\overline{U}_1y+\bar{\lambda}_1,\ \overline{\Phi}_{20}=-\overline{U}_2y+\bar{\lambda}_2\ , \tag{3.2}$$

where "$\overline{(\ )}$" denotes the zonal mean, $\overline{U}_i$ are constants, and $\bar{\lambda}_i$ are constants of integration. In the present context, for convenience, it is supposed that $\bar{\lambda}_2=0$, which does not affect the final results. The domain Ω under consideration is a zonal periodic channel

$$-\pi<x<\pi\ ,\quad 0<y<l\ .$$

The potential vorticity of the initial zonal mean flow is denoted by $\overline{P}_{i0}$. According to the criterion for the Phillips model in section 2, when the initial zonal mean flow $(\overline{\Phi}_{i0}, \overline{P}_{i0})$ is located in regions I and II (see Fig. 1), it is possibly nonlinearly unstable. Now, the aim is to derive upper bounds on the energy $E_{\Phi'_i}$ of the wavy (nonzonal) part $\Phi'_i = \Phi_i - \overline{\Phi}_i$ at any time, where the expression of $E_{\Phi'_i}$ is the same as $E(t)$, but with Φ'_i instead of ψ_i. We take $d = 1$ throughout the following.

3.1 *The case of region I*

First, we look at the case of the initial zonal mean flow $(\overline{\Phi}_{i0}, \overline{P}_{i0})$ being in region I, i.e.,

$$\frac{F\overline{U}_s}{\beta} > 1, \quad 0 < \frac{\lambda}{F} \leqslant \sqrt{2} \ . \tag{3.3}$$

Suppose that

$$\frac{F\overline{U}_s}{\beta} = 1 + \varepsilon \ , \tag{3.4}$$

where $\varepsilon > 0$ is a supercriticality.

Choose in region III the artificial stable basic flow $(\Psi_i(y), Q_i(y))$, satisfying

$$U_i = -\frac{d\Psi_i}{dy} \ , \quad \frac{FU_s}{\beta} = 1 - \delta \ , \tag{3.5}$$

where the parameter $\delta (0 < \delta \leqslant 1)$ is introduced to seek the following minimum. The stream functions for the artificial stable basic flow are

$$\Psi_i = -U_i y + \lambda_i \ , \quad i = 1, 2, \tag{3.6}$$

where λ_i are arbitrary constants. It is also supposed that $\lambda_2 = 0$. Decompose Φ_i into two parts:

$$\Phi_i = \Psi_i + \psi_i \ , \tag{3.7}$$

where ψ_i is the disturbance relative to the artificial basic flow Ψ_i, with the initial value

$$\psi_{10} = (U_1 - \overline{U}_1)y + \Phi'_{10} - \lambda_1 \ , \tag{3.8}$$

$$\psi_{20} = (U_2 - \overline{U}_2)y + \Phi'_{20} \ . \tag{3.9}$$

In (3.8), $\lambda_1 - \bar{\lambda}_1$ is now denoted by λ_1. Then, for ψ_i, (2.18) holds. By orthogonality, it follows from (2.18) that

$$E_{\Phi'_i} \leqslant E(0) + \frac{U_s}{2(\beta - FU_s)} \int_\Omega q'^2_{20}\, dxdy \equiv E_B \ . \tag{3.10}$$

Now a calculation of E_B is made. We first calculate the second term on the right side of the inequality in (3.10).

$$q_{20} = \nabla^2 \psi_{20} - F(\psi_{20} - \psi_{10}) = \nabla^2 \Phi'_{20} - F(\Phi'_{20} - \Phi'_{10}) - F[(\overline{U}_s - U_s)y + \lambda_1] \ ,$$

$$q_2^* = \frac{\int_\Omega q_{20}\, dxdy}{\int_\Omega dxdy} = -F\left[\lambda_1 + \frac{l(\overline{U}_s - U_s)}{2}\right] .$$

$$\int_\Omega \frac{1}{2} q_{20}^2 \, dxdy = T_2 + \frac{1}{12}\pi F^2 l^3 (\overline{U}_s - U_s)^2 + \pi F^2 l \left[\lambda_1 + \frac{l(\overline{U}_s - U_s)}{2} \right]^2 ,$$

$$\int_\Omega \frac{1}{2} q_2^{*2} \, dxdy = \pi l q_2^{*2} = \pi l F^2 \left[\lambda_1 + \frac{l(\overline{U}_s - U_s)}{2} \right]^2 ,$$

where

$$T_2 = \int_\Omega \frac{1}{2} [\nabla^2 \Phi'_{20} - F(\Phi'_{20} - \Phi'_{10})]^2 \, dxdy .$$

According to (2.16),

$$\int_\Omega \frac{1}{2} q'^2_{20} \, dxdy = T_2 + \frac{1}{12}\pi F^2 l^3 (\overline{U}_s - U_s)^2 ,$$

which, clearly, is independent of λ_1.

Next we calculate $E(0)$.

$$\begin{aligned} E(0) &= \int_\Omega \frac{1}{2} [|\nabla\psi_{10}|^2 + |\nabla\psi_{20}|^2 + F(\psi_{20} - \psi_{10})^2] dxdy \\ &= T_0 + \pi\int_0^l \{(U_1 - \overline{U}_1)^2 + (U_2 - \overline{U}_2)^2 + F[(\overline{U}_s - U_s)y + \lambda_1]^2\} dy \\ &= T_0 + \pi l[(U_1 - \overline{U}_1)^2 + (U_2 - \overline{U}_2)^2] + \pi F l [\lambda_1^2 + \lambda_1 l(\overline{U}_s - U_s) \\ &\quad + \frac{1}{3} l^2 (\overline{U}_s - U_s)^2] , \end{aligned}$$

where $T_0 = \int_\Omega \frac{1}{2} [|\nabla\Phi'_{10}|^2 + |\nabla\Phi'_{20}|^2 + F(\Phi'_{20} - \Phi'_{10})^2] dxdy$.

Note that

$$\begin{aligned} &(U_1 - \overline{U}_1)^2 + (U_2 - \overline{U}_2)^2 \\ &= \left[\frac{U_1 + U_2}{2} + \frac{U_1 - U_2}{2} - \left(\frac{\overline{U}_1 + \overline{U}_2}{2} + \frac{\overline{U}_1 - \overline{U}_2}{2} \right) \right]^2 \\ &\quad + \left[\frac{U_1 + U_2}{2} - \frac{U_1 - U_2}{2} - \left(\frac{\overline{U}_1 + \overline{U}_2}{2} - \frac{\overline{U}_1 - \overline{U}_2}{2} \right) \right]^2 \\ &= \left(\frac{U_1 + U_2}{2} - \frac{\overline{U}_1 + \overline{U}_2}{2} + \frac{U_s - \overline{U}_s}{2} \right)^2 + \left(\frac{U_1 + U_2}{2} - \frac{\overline{U}_1 + \overline{U}_2}{2} - \frac{U_s - \overline{U}_s}{2} \right)^2 \\ &= \frac{(U_1 + U_2 - \overline{U}_1 - \overline{U}_2)^2}{2} + \frac{(U_s - \overline{U}_s)^2}{2} , \end{aligned} \tag{3.12}$$

and

$$\lambda_1^2 + \lambda_1 l(\overline{U}_s - U_s) + \frac{1}{3} l^2 (\overline{U}_s - U_s)^2 = \left[\lambda_1 + \frac{l(\overline{U}_s - U_s)}{2} \right]^2 + \frac{1}{12} l^2 (\overline{U}_s - U_s)^2 . \tag{3.13}$$

so, when

$$U_1 + U_2 - \overline{U}_1 - \overline{U}_2 = 0 \tag{3.14}$$

and

$$\lambda_1 = -\frac{l(\overline{U}_s - U_s)}{2} , \tag{3.15}$$

$E(0)$ has the minimum

$$E(0) = T_0 + \pi l (\overline{U}_s - U_s)^2 \left(\frac{1}{2} + \frac{F l^2}{12} \right) . \tag{3.16}$$

Utilizing (3.11) and (3.16),

$$E_B = T_0 - \frac{T_2}{F} + \frac{I(\delta)}{F} \ , \tag{3.17}$$

where

$$I(\delta) = \pi l \beta^2 \left(\frac{1}{2F} + \frac{l^2}{12\delta}\right)(\varepsilon + \delta)^2 + \frac{T_2}{\delta} \ . \tag{3.18}$$

For infinitesimal initial disturbances, i.e., $T_0 \to 0$, $T_2 \to 0$, when

$$\delta_+ = \frac{Fl^2}{24}\left[-1 + \left(1 + \frac{48\varepsilon}{Fl^2}\right)^{1/2}\right] , \tag{3.19}$$

which automatically satisfies $0 < \delta \leqslant 1$, E_B reaches the minimum (also denoted by E_B).

Now that the value of E_B at $\delta = 1$ is exactly the total amount of energy for the system $E_T(t) = T_0 + \pi l\left(\frac{1}{2} + \frac{Fl^2}{12}\right)\overline{U}_s^2$, clearly, the value of E_B at $\delta = \delta_+$ is less than the total amount of energy for the system.

Shepherd (1993) studied the nonlinear saturation of baroclinic instability in a general case, which, for $D_1 = 1$, $F_1 = F_2 = F$, reduces to the present case (for $l = 1$). It is easy to see that for $l = 1$, Eq.(3.17) is less than 2π times Eq.(4.16) in Shepherd (1993) for $D_1 = 1$, $F_1 = F_2 = F$. (Since the energy in this paper is integrated over Ω with a length 2π in the x direction, while in Shepherd (1993) it is integrated over y and averaged over x, then, for E_B and his (4.16) to be comparable, E_B must be divided by 2π or his (4.16) is multiplied by 2π).

3.2 *The case of region II*

Now we consider the initial zonal mean flow ($\overline{\Phi}_{i0}$, $\overline{P}_{i0}$) being in region II, i.e.,

$$\frac{F\overline{U}_s}{\beta} > U_c \ , \quad \sqrt{2} < \frac{\lambda}{F} < 2 \ , \tag{3.20}$$

a case which, in fact, was not studied definitely by Shepherd (1993).

We still set

$$\frac{F\overline{U}_s}{\beta} = U_c(1 + \varepsilon) \ , \tag{3.21}$$

where $\varepsilon > 0$ is a supercriticality.

Choose a class of artificial stable basic flows ($\Psi_i(y)$, $Q_i(y)$), satisfying

$$U_i = -\frac{d\Psi_i}{dy} \ , \tag{3.22}$$

$$\frac{FU_s}{\beta} = U_c(1 - \delta), \quad \sqrt{2} < \frac{\lambda}{F} < 2 \ , \tag{3.23}$$

where U_i are constants, $U_s = U_1 - U_2$, $1 - 1/U_c < \delta \leqslant 1$ is a parameter, which is chosen so that

$$0 \leqslant \frac{FU_s}{\beta} < 1 \ . \tag{3.24}$$

Therefore, the flow ($\Psi_i(y)$, $Q_i(y)$) is located in region III.

Just as in Subsection 3.1, one has

$$E_{\Phi'_i} \leqslant E(0)+ \frac{U_s}{2(\beta- FU_s)}\int_{\Omega} q'^2_{20}dxdy \equiv E_B \ , \tag{3.25}$$

and

$$E_B = T_0 + \frac{U_c(1-\delta)}{F[1-U_c(1-\delta)]}T_2 + \frac{\pi l\beta^2 U_c^2}{F^2}\left[\frac{1}{2}+\frac{Fl^2}{12}\frac{1}{1-U_c(1-\delta)}\right](\varepsilon+\delta)^2 \ , \tag{3.26}$$

where the expressions of T_0 and T_2 are the same as above.

We consider the case of Φ'_{i0} being an infinitesimal disturbance, i.e., $T_0 \to 0, T_2 \to 0$. Then,

$$E_B = \frac{\pi l\beta^2 U_c^2}{F}(\varepsilon+\delta)^2\left[\frac{1}{2F}+\frac{l^2}{12}\frac{1}{1-U_c(1-\delta)}\right] . \tag{3.27}$$

It is provable that the minimum of E_B (also denoted by E_B) will be attained at

$$\delta_+ = 1-\frac{1}{U_c}+\frac{2(1+\varepsilon-\frac{1}{U_c})}{1+\left[1+\frac{48U_c}{Fl^2}\left(1+\varepsilon-\frac{1}{U_c}\right)\right]^{1/2}} \tag{3.28}$$

for

$$\sqrt{2}<\frac{\lambda}{F}<1.98, \quad 0<\varepsilon\leqslant\frac{2}{U_c}\left(1+\frac{6}{Fl^2}\right)-1 \ . \tag{3.29}$$

Noting that the value of E_B at $\delta = 1$ is exactly the total energy E_T of the system, it is obvious that

$$E_B \leqslant E_T \ , \tag{3.30}$$

which offers an effective constraint on wavy (nonzonal) disturbances Φ'_i at any time.

4. Comparison

4.1 *The case of region I*

Shepherd (1993), in the context of Arnold's first theorem, studied the nonlinear saturation of baroclinic instability in the Phillips model, and obtained in a general case bounds on the energy, which, for $D_1 = 1$, $F_1 = F_2 = F$, is just the present case (for $l = 1$). Since, for $l = 1$, Eq.(3.17) is less than 2π times Eq.(4.16) in Shepherd (1993) for $D_1 = 1$, $F_1 = F_2 = F$, obviously, the corresponding optimal E_B here over the interval $0<\delta\leqslant 1$ is tighter than 2π times the optimal upper bound of Shepherd (namely, the local minimum of (4.16)). According to (3.19), for infinitesimal initial disturbances Φ'_{i0}, i.e., $T_0 \to 0, T_2 \to 0$, when $\varepsilon \ll Fl^2/48$, δ_+ is well approximated by $\delta_+ \approx \varepsilon$. So

$$E_B \approx \frac{\pi l\beta^2}{F^2}\left(2\varepsilon+\frac{Fl^2}{3}\right)\varepsilon \ . \tag{4.1}$$

In particular, for $l = 1$, Eq.(4.1) is

$$E_{\rm B} \approx \frac{\pi\beta^2}{F^2}\left(2\varepsilon + \frac{F}{3}\right)\varepsilon \ , \tag{4.2}$$

which, when divided by 2π, is less than Eq.(4.20) in Shepherd (1993) for $D_1 = 1$, $F_1 = F_2 = F$, i.e.,

$$\frac{\beta^2}{F^2}\left(2\varepsilon + \frac{F}{6}\right)\varepsilon \ . \tag{4.3}$$

Noting that the leading terms (i.e. the first–order terms in ε) in $E_{\rm B} / 2\pi$ and (4.3) are the same, and that the discrepancy lies in the second–order terms, we conclude that the result obtained here is a second–order correction in ε to Shepherd's.

On the other hand, XL, for the case of region I, using Arnold's second stability theorem, studied the nonlinear saturation of baroclinic instability and obtained the upper bound on the potential enstrophy of wavy disturbances, which, for infinitesimal initial disturbances Φ'_{i0}, is

$$Z_{\rm m} = \frac{2\pi\varepsilon\beta^2 l^3}{3} \ . \tag{4.4}$$

Hence, the upper bound on the energy of wavy disturbances derived from (4.4) through the Poincare inequality is

$$E_{\rm m} = \frac{Z_{\rm m}}{\lambda} \ . \tag{4.5}$$

The ratio of (4.1) and (4.5) is

$$\frac{E_{\rm B}}{E_{\rm m}} = \frac{\pi^2}{Fl^2}\left(\frac{3\varepsilon}{Fl^2} + \frac{1}{2}\right) \to \frac{\pi^2}{2Fl^2} = \frac{\lambda}{2F}, \quad \text{as} \quad \varepsilon \to 0 \ . \tag{4.6}$$

In view of (3.3), it is clear that

$$0 < \frac{\lambda}{2F} \leqslant \frac{1}{\sqrt{2}} < 1 \ . \tag{4.7}$$

Therefore, for sufficiently small ε, $E_{\rm B}$ is less than $E_{\rm m}$. Especially, when $\lambda / F \ll \sqrt{2}$, $E_{\rm B}$ is evidently superior to $E_{\rm m}$. Although (4.6) and (4.7) are similar to (4.23) in Shepherd (1993), the former hold in a general case.

Figure 2 displays $E_{\rm B}$, $2\pi E_{\rm s}$ (Shepherd's result), $E_{\rm m}$, and $E_{\rm T}$ (the total energy of the system) as functions of ε for $d = 1$, $l = 1$, $\beta = 10$, $F = 10$ (i.e., $\lambda / F \approx 0.987$), and infinitesimal initial disturbances Φ'_{i0}.

4.2 *The case of region II*

This case is not studied definitely in Shepherd's paper (1993). So, a comparison is made here between the upper bound given above and that derived from the upper bound on the potential enstrophy (XL) through the Poincare inequality.

In the present case, for infinitesimal initial disturbances Φ'_{i0}, XL gives the upper bound on the potential enstrophy of wavy disturbances as

$$Z_{\rm m} = \frac{2\pi\beta^2 l^3}{3}[U_c(1+\varepsilon) - 1] \ , \tag{4.8}$$

for
$$\sqrt{2} < \frac{\lambda}{F} < \frac{\sqrt{6}+\sqrt{2}}{2} \approx 1.93, \quad \varepsilon \leqslant \frac{2}{U_c} - 1 \ . \tag{4.9}$$

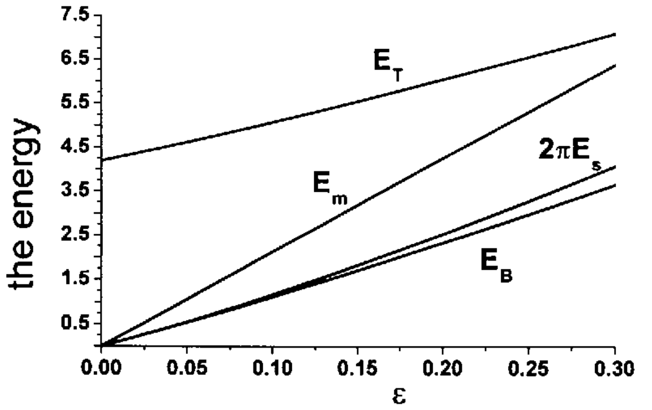

Fig. 2. E_B (i.e., Eq.(3.17)), $2\pi E_s$ (Shepherd's result), E_m, and E_T (the total energy of the system) are functions of ε for $d=1$, $l=1$, $\beta=10$, $F=10$ (i.e., $\lambda/F\approx 0.987$), and infinitesimal initial disturbances Φ'_{i0}.

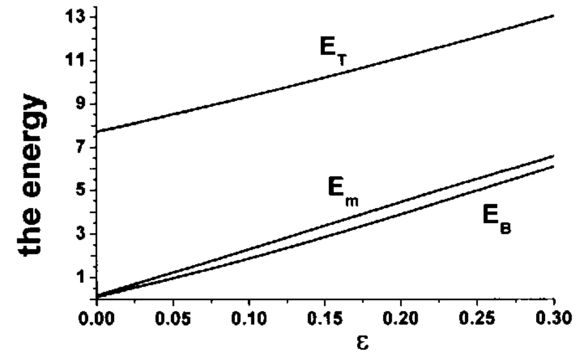

Fig. 3. E_B (i.e., Eq.(3.27)), E_m and E_T (the total energy of the system) are functions of ε for $d=1$, $l=1$, $\beta=10$, $F=6.58$ (i.e., $\lambda/F\approx 1.5$), and infinitesimal initial disturbances Φ'_{i0}.

Then, the upper bound on the energy of wavy disturbances through the Poincare inequality is

$$E_m=\frac{Z_m}{\lambda}=2\pi l\beta^2[U_c(1+\varepsilon)-1]\cdot\frac{l^4}{3\pi^2}\ . \tag{4.10}$$

On the other hand, according to (3.28), when

$$\frac{48}{Fl^2}[U_c(1+\varepsilon)-1]\ll 1\ ,$$

i.e.

$$\varepsilon\ll\frac{1}{U_c}\left(1+\frac{Fl^2}{48}\right)-1\ , \tag{4.11}$$

then δ_+ is well approximated as

$$\delta_+ \approx 2+\varepsilon-\frac{2}{U_c}\ , \tag{4.12}$$

which makes

$$E_B \approx 2\pi l\beta^2[U_c(1+\varepsilon)-1]\left[\frac{U_c(1+\varepsilon)-1}{F^2}+\frac{l^2}{6F}\right]. \tag{4.13}$$

It is straightforward to prove that, when

$$\frac{\lambda}{F}<1.70, \quad \text{and} \quad \varepsilon \ll \frac{1}{U_c}\left(1+\frac{Fl^2}{48}\right)-1\ , \tag{4.14}$$

one has

$$\frac{(4.13)}{(4.10)}=\frac{3[U_c(1+\varepsilon)-1]}{\pi^2}\left(\frac{\lambda}{F}\right)^2+\frac{1}{2}\cdot\frac{\lambda}{F}\to\frac{\sqrt{2}}{2}<1, \quad \text{as} \quad \frac{\lambda}{F}\to\sqrt{2},\ \varepsilon\to 0\ . \tag{4.15}$$

In consequence, for sufficiently small ε, and for λ/F sufficiently close to $\sqrt{2}$, E_B is less than E_m.

Figure 3 displays E_B, E_m, and E_T (the total energy of the system) as functions of ε for $d=l$, $l=1$, $\beta=10$, $F=6.58$ (i.e., $\lambda/F\approx 1.5$), and infinitesimal initial disturbances Φ'_{i0}.

5. Conclusions and discussion

In this paper, a conservation law for the Phillips model is first derived. Then, the nonlinear saturation of a purely baroclinic instability caused by the vertical velocity shear of the basic flow in the Phillips model is studied within the context of Arnold's second stability theorem. For region I (see Fig. 1), our result is a second–order correction in ε to Shepherd's value. For region II, the analytic upper bound on the energy of wavy disturbances offers an effective constraint on wavy (nonzonal) disturbances Φ'_i at any time.

There is one interesting problem. It is known from Fig. 1 that the stability of the basic flow in the Phillips model is determined only by FU_s/β and λ/F, and the saturation bounds on the energy of wavy disturbances are also associated with λ/F (for region I, $0<\lambda/F\leqslant\sqrt{2}$; for region II, $\sqrt{2}<\lambda/F<1.98$). But if we let l (i.e., λ) and F change simultaneously while keeping the values of λ/F and other parameters fixed, then, how do the saturation upper bounds E_B change with l? For the case of region I, consider infinitesimal initial disturbances Φ'_{i0}. Noting that (5.14) and δ_+ are dependent only on λ/F, it can be shown that $E_B\sim l^5$. This conclusion also holds for the case of region II, and infinitesimal initial disturbances Φ'_{i0}.

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Phillips 模式的斜压不稳定的非线性饱和问题: 能量情形

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摘　要

首先推导了 Phillips 模式的一个守衡律，利用它，在 Arnold 第二定理的范围内研究了 Phillips 模式中基流的垂直速度切变引起的斜压不稳定的非线性饱和问题，得到了波动扰动能量的解析上界估计；对参数平面上的一个不稳定区域，我们的结果是对 Shepherd 结果的二阶修正（关于超临界数），对另一个不稳定区域，我们的结果提供了一个对任意时刻波动扰动Φ'_i的有效约束。

关键词: 非线性饱和，斜压不稳定，Phillips 模式