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A Comparative Study of Conservative and Nonconservative Schemes

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ABSTRACT

For the conservative and non-conservative schemes of nonlinear evolution equations, by taking the two-dimensional shallow water wave equations as an example, a comparative analysis on computational stability is carried out. The relationship between the nonlinear computational stability, the structure of the difference schemes, and the form of initial values is also discussed.

Key words: conservative scheme, nonconservative scheme, computational stability, initial value

1. Introduction

In medium and long-range numerical weather prediction and ocean current numerical simulation, the finite-difference schemes are mostly employed to carry out the numerical solutions of nonlinear atmospheric and oceanic equations. So it is a key problem on how to design long-time computationally stable difference schemes. Zeng (1978), Zeng and Ji (1981), and Ji (1981a, b) systematically studied the computational stability of the adiabatic or non-dissipative nonlinear evolution equations, discussed the reasons causing nonlinear computational instability, and constructed a computationally stable implicit complete square conservative difference scheme. Later, Ji and Wang (1990; 1991; 1994; 1995) and Ji et al. (1998) constructed a computationally stable explicit complete square conservative difference scheme. Furthermore, Wang et al. (1995) also discussed the relationship between the square conservation systems and Hamiltonian systems. For the nonconservative schemes of the nonlinear evolution equations, Lin et al. (2000) also gave a new method for judging the computational stability. Meanwhile, Lin et al. (2002) carried out a comparative analysis on the computational stability for linear and nonlinear evolution equations and proved that the computational stability of the difference schemes of nonlinear evolution equations is totally different from that of the linear evolution equations. In this paper, taking the two-dimensional shallow water wave equations as an example, a comparative analysis on computational stability is carried out for conservative and nonconservative schemes, and the relationships among the nonlinear computational stability, the construction of conservative and nonconservative schemes, and the form of initial values is further discussed.

2. The square conservative scheme and its computational stability

The two-dimensional nonlinear shallow water wave equations are

$$\begin{cases} \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+\dfrac{\partial \varphi}{\partial x}=0\,, \\ \dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+\dfrac{\partial \varphi}{\partial y}=0\,, \\ \dfrac{\partial \varphi}{\partial t}+u\dfrac{\partial \varphi}{\partial x}+v\dfrac{\partial \varphi}{\partial y}+(\Phi+\varphi)\left(\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)=0\,, \end{cases} \tag{1}$$

where $\varphi=g\xi, \Phi=gh$, ξ is surface elevation, g is gravitational acceleration, and h is water depth. g and h are constants.

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Equation (1) can be rewritten in an operator form,

$$\frac{\partial \boldsymbol{F}}{\partial t}+\boldsymbol{L}\boldsymbol{F}=0\,, \tag{2}$$

where

$$\boldsymbol{F}=\begin{pmatrix} u \\ v \\ \phi \end{pmatrix},$$

$$\boldsymbol{L}=\begin{pmatrix} u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} \\ 0 & u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ (\Phi+\varphi)\frac{\partial}{\partial x} & (\Phi+\varphi)\frac{\partial}{\partial y} & u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y} \end{pmatrix}.$$

For Eq. (2), we take C grids and use the difference scheme as follows:

Scheme 1

$$\boldsymbol{F}_{i,j}^{n+1}=\boldsymbol{F}_{i,j}^{n}-\Delta t\boldsymbol{A}\boldsymbol{F}_{i,j}^{n}-\varepsilon\Delta t^2\boldsymbol{B}\boldsymbol{F}_{i,j}^{n}\,, \tag{3}$$

where

$$\boldsymbol{F}_{i,j}^{n+1}=\begin{pmatrix} u_{i+1/2,j}^{n+1} \\ v_{i,j+1/2}^{n+1} \\ \varphi_{i,j}^{n+1} \end{pmatrix},\quad \boldsymbol{F}_{i,j}^{n}=\begin{pmatrix} u_{i+1/2,j}^{n} \\ v_{i,j+1/2}^{n} \\ \varphi_{i,j}^{n} \end{pmatrix},$$

$$\boldsymbol{A}\boldsymbol{F}_{i,j}^{n}=\begin{pmatrix} \frac{u_{i+3/2,j}^n-u_{i+1/2,j}^n}{\Delta x} & \frac{u_{i+1/2,j+1}^n-u_{i+1/2,j}^n}{\Delta y} & \frac{\varphi_{i+1,j}^n-\varphi_{i,j}^n}{\Delta x} & 0 \\ \frac{v_{i+1,j+1/2}^n-v_{i,j+1/2}^n}{\Delta x} & \frac{v_{i,j+3/2}^n-v_{i,j+1/2}^n}{\Delta x} & \frac{\varphi_{i,j+1}^n-\varphi_{i,j}^n}{\Delta y} & 0 \\ \frac{\varphi_{i+1,j}^n-\varphi_{i,j}^n}{\Delta x} & \frac{\varphi_{i,j+1}^n-\varphi_{i,j}^n}{\Delta y} & 0 & U_{i,j}^n \end{pmatrix}\begin{pmatrix} u_{i+1/2,j}^n \\ v_{i,j+1/2}^n \\ 1 \\ \Phi+\varphi_{i,j}^n \end{pmatrix},$$

$$U_{i,j}^n=\frac{u_{i+3/2,j}^n-u_{i+1/2,j}^n}{\Delta x}+\frac{v_{i,j+3/2}^n-v_{i,j+1/2}^n}{\Delta y}\,,$$

ε is a dissipative coefficient, $\boldsymbol{B}$ is a dissipative operator, and their concrete meanings are according to Ji and Wang (1991). When ε and $\boldsymbol{B}$ are suitably selected, scheme 1 can become a completely square conservation.

According to Ji and Wang (1991) and Wang and Ji (1990; 1994; 1995), the following theorem can be proved easily.

Theorem 1 If Δt is fixed, and Δt satisfies $2\Delta t\sqrt{K_1K_3}<1$, then scheme 1 is an explicit complete square conservative scheme with a constant time-step while

$$\varepsilon=\frac{K_1}{1-\Delta tK_2+\sqrt{(1-\Delta tK_2)^2-\Delta t^2K_1K_3}}\,, \tag{4}$$

where

$$\begin{cases} K_1=\dfrac{\left\|\boldsymbol{A}\boldsymbol{F}_{i,j}^n\right\|^2}{\left(\boldsymbol{B}\boldsymbol{F}_{i,j}^n,\boldsymbol{F}_{i,j}^n\right)}\,, \\ K_2=\dfrac{\left(\boldsymbol{A}\boldsymbol{F}_{i,j}^n,\boldsymbol{B}\boldsymbol{F}_{i,j}^n\right)}{\left(\boldsymbol{B}\boldsymbol{F}_{i,j}^n,\boldsymbol{F}_{i,j}^n\right)}\,, \\ K_3=\dfrac{\left\|\boldsymbol{B}\boldsymbol{F}_{i,j}^n\right\|^2}{\left(\boldsymbol{B}\boldsymbol{F}_{i,j}^n,\boldsymbol{F}_{i,j}^n\right)}\,, \end{cases} \tag{5}$$

$$\boldsymbol{B}\boldsymbol{F}_{i,j}^n=\frac{\boldsymbol{A}\boldsymbol{F}_{i,j}^{n+1}-\boldsymbol{A}\boldsymbol{F}_{i,j}^n}{\Delta t}\,,\quad \tilde{\boldsymbol{F}}_{i,j}^{n+1}=\boldsymbol{F}_{i,j}^n-\Delta t\boldsymbol{A}\boldsymbol{F}_{i,j}^n\,. \tag{6}$$

Theorem 2 If the τ is not fixed, while $K_1 < 2\varepsilon$ and

$$\Delta t = \frac{2 - \frac{1}{\varepsilon}K_1}{K_2 + \sqrt{K_2^2 + \varepsilon\left(2 - \frac{1}{\varepsilon}K_1\right)K_3}}\,, \tag{7}$$

then scheme 1 is an explicit complete square conservative scheme with an adjustable time-step, where K_1, K_2, and K_3 are determined by Expression (5), ε by Expression (4), and $\boldsymbol{B}$ by Expression (6).

3. The nonconservative scheme and its computational stability

For Eq. (1), we take C grids and use two kinds of nonconservative difference schemes as follows:

Scheme 2 (CTCS scheme)

$$\frac{u^{n+1}_{i+1/2,j} - u^{n-1}_{i+1/2,j}}{2\Delta t} + u^n_{i+1/2,j}\frac{u^n_{i+3/2,j} - u^n_{i-1/2,j}}{2\Delta x} + v^n_{i,j+1/2}\frac{u^n_{i+1/2,j+1} - u^n_{i+1/2,j-1}}{2\Delta y} + \frac{\varphi^n_{i+1,j} - \varphi^n_{i-1,j}}{2\Delta x} = 0\,. \tag{8}$$

$$\frac{v^{n+1}_{i,j+1/2} - v^{n-1}_{i,j+1/2}}{2\Delta t} + u^n_{i+1/2,j}\frac{v^n_{i+1,j+1/2} - v^n_{i-1,j+1/2}}{2\Delta x} + v^n_{i,j+1/2}\frac{v^n_{i,j+3/2} - v^n_{i,j-1/2}}{2\Delta y} + \frac{\varphi^n_{i,j+1} - \varphi^n_{i,j-1}}{2\Delta y} = 0\,. \tag{9}$$

$$\frac{\varphi^{n+1}_{i,j} - \varphi^{n-1}_{i,j}}{2\Delta t} + u^n_{i+1/2,j}\frac{\varphi^n_{i+1,j} - \varphi^n_{i-1,j}}{2\Delta x} + v^n_{i,j+1/2}\frac{\varphi^n_{i,j+1} - \varphi^n_{i,j-1}}{2\Delta y} + \left(\Phi + \varphi^n_{i,j}\right)\left(\frac{u^n_{i+3/2,j} - u^n_{i-1/2,j}}{2\Delta x} + \frac{v^n_{i,j+3/2} - v^n_{i,j-1/2}}{2\Delta y}\right) = 0\,. \tag{10}$$

Scheme 3 (Lax-Wendroff scheme)

$$\begin{aligned}
&\frac{u^{n+1}_{i+1/2,j} - u^{n}_{i+1/2,j}}{\Delta t} + u^n_{i+1/2,j}\frac{u^n_{i+3/2,j} - u^n_{i-1/2,j}}{2\Delta x} + v^n_{i,j+1/2}\frac{u^n_{i+1/2,j+1} - u^n_{i+1/2,j-1}}{2\Delta y} + \frac{\varphi^n_{i+1,j} - \varphi^n_{i-1,j}}{2\Delta x} \\
&= \frac{\left[\left(u^n_{i+1/2,j}\right)^2 + \Phi + \varphi^n_{i,j}\right]\Delta t}{2\Delta x^2}\left(u^n_{i+3/2,j} - 2u^n_{i+1/2,j} + u^n_{i-1/2,j}\right) + \frac{v^n_{i,j+1/2}\Delta t}{2\Delta x\Delta y}\left(\varphi^n_{i+1,j+1} - 2\varphi^n_{i,j} + \varphi^n_{i-1,j-1}\right) \\
&+ \frac{u^n_{i+1/2,j}v^n_{i,j+1/2}\Delta t}{2\Delta x\Delta y}\left(u^n_{i+3/2,j} - 2u^n_{i+1/2,j} + u^n_{i-1/2,j-1}\right) + \frac{\left(v^n_{i,j+1/2}\right)^2\Delta t}{2\Delta y^2}\left(u^n_{i+1/2,j+1} - 2u^n_{i+1/2,j} + u^n_{i+1/2,j-1}\right) \\
&+ \frac{\left(\Phi + \varphi^n_{i,j}\right)\Delta t}{4\Delta x\Delta y}\left(v^n_{i+1,j+3/2} - 2v^n_{i,j+1/2} + v^n_{i-1,j-1/2}\right) + \frac{u^n_{i+1/2,j}\Delta t}{\Delta x^2}\left(\varphi^n_{i+1,j} - 2\varphi^n_{i,j} + \varphi^n_{i-1,j}\right).
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\frac{v^{n+1}_{i,j+1/2} - v^{n}_{i,j+1/2}}{\Delta t} + u^n_{i+1/2,j}\frac{v^n_{i+1,j+1/2} - v^n_{i-1,j+1/2}}{2\Delta x} + v^n_{i,j+1/2}\frac{v^n_{i,j+3/2} - v^n_{i,j-1/2}}{2\Delta y} + \frac{\varphi^n_{i,j+1} - \varphi^n_{i,j-1}}{2\Delta y} \\
&= \frac{\left(u^n_{i+1/2,j}\right)^2\Delta t}{2\Delta x^2}\left(v^n_{i+1,j+1/2} - 2v^n_{i,j+1/2} + v^n_{i-1,j+1/2}\right) + \frac{u^n_{i+1/2,j}v^n_{i,j+1/2}\Delta t}{2\Delta x\Delta y}\left(v^n_{i+1,j+3/2} - 2v^n_{i,j+1/2} + v^n_{i-1,j-1/2}\right) \\
&+ \frac{\left[\left(v^n_{i,j+1/2}\right)^2 + \Phi + \varphi^n_{i,j}\right]\Delta t}{2\Delta y^2}\left(v^n_{i,j+3/2} - 2v^n_{i,j+1/2} + v^n_{i,j-1/2}\right) + \frac{u^n_{i+1/2,j}\Delta t}{2\Delta x\Delta y}\left(\varphi^n_{i+1,j+1} - 2\varphi^n_{i,j} + \varphi^n_{i-1,j-1}\right)
\end{aligned}$$

$$+\frac{\left(\Phi+\varphi_{i,j}^{n}\right)\Delta t}{4\Delta x\Delta y}\left(u_{i+3/2,j+1}^{n}-2u_{i+1/2,j}^{n}+u_{i-1/2,j-1}^{n}\right)+\frac{v_{i,j+1/2}^{n}\Delta t}{\Delta y^{2}}\left(\varphi_{i,j+1}^{n}-2\varphi_{i,j}^{n}+\varphi_{i,j-1}^{n}\right). \tag{12}$$

$$\begin{aligned}
\frac{\varphi_{i,j}^{n+1}-\varphi_{i,j}^{n}}{\Delta t}&+u_{i+1/2,j}^{n}\frac{\varphi_{i+1,j}^{n}-\varphi_{i-1,j}^{n}}{2\Delta x}+v_{i,j+1/2}^{n}\frac{\varphi_{i,j+1}^{n}-\varphi_{i,j-1}^{n}}{2\Delta y}\\
&+\left(\Phi+\varphi_{i,j}^{n}\right)\left(\frac{u_{i+3/2,j}^{n}-u_{i-1/2,j}^{n}}{2\Delta x}+\frac{v_{i,j+3/2}^{n}-v_{i,j-1/2}^{n}}{2\Delta y}\right)\\
&=\frac{u_{i+1/2,j}^{n}\left(\Phi+\varphi_{i,j}^{n}\right)\Delta t}{\Delta x^{2}}\left(u_{i+3/2,j}^{n}-2u_{i+1/2,j}^{n}+u_{i-1/2,j}^{n}\right)\\
&+\frac{v_{i,j+1/2}^{n}\left(\Phi+\varphi_{i,j}^{n}\right)\Delta t}{2\Delta x\Delta y}\left(u_{i+3/2,j+1}^{n}-2u_{i+1/2,j}^{n}+u_{i-1/2,j-1}^{n}\right)\\
&+\frac{u_{i+1/2,j}^{n}\left(\Phi+\varphi_{i,j}^{n}\right)\Delta t}{2\Delta x\Delta y}\left(v_{i+1,j+3/2}^{n}-2v_{i,j+1/2}^{n}+v_{i-1,j-1/2}^{n}\right)\\
&+\frac{v_{i,j+1/2}^{n}\left(\Phi+\varphi_{i,j}^{n}\right)\Delta t}{\Delta y^{2}}\left(v_{i,j+3/2}^{n}-2v_{i,j+1/2}^{n}+v_{i,j-1/2}^{n}\right)\\
&+\frac{\left[\left(u_{i+1/2,j}^{n}\right)^{2}+\Phi+\varphi_{i,j}^{n}\right]\Delta t}{2\Delta x^{2}}\left(\varphi_{i+1,j}^{n}-2\varphi_{i,j}^{n}+\varphi_{i-1,j}^{n}\right)\\
&+\frac{u_{i+1/2,j}^{n}v_{i,j+1/2}^{n}\Delta t}{2\Delta x\Delta y}\left(\varphi_{i+1,j+1}^{n}-2\varphi_{i,j}^{n}+\varphi_{i-1,j-1}^{n}\right)\\
&+\frac{\left[\left(v_{i,j+1/2}^{n}\right)^{2}+\Phi+\varphi_{i,j}^{n}\right]\Delta t}{2\Delta y^{2}}\left(\varphi_{i,j+1}^{n}-2\varphi_{i,j}^{n}+\varphi_{i,j-1}^{n}\right).
\end{aligned} \tag{13}$$

According to Lin et al. (2000), the following theorem can be easily proved.

Theorem 3 For schemes 2 and 3, the necessary conditions for computational stability are

(i) $u(x,y,0)>0, v(x,y,0)>0$;

(ii) $\dfrac{\partial u(x,y,0)}{\partial x}\geqslant 0, \dfrac{\partial u(x,y,0)}{\partial y}\geqslant 0$;

(iii) $\dfrac{\partial v(x,y,0)}{\partial x}\geqslant 0, \dfrac{\partial v(x,y,0)}{\partial y}\geqslant 0$;

(iv) $\dfrac{\partial \varphi(x,y,0)}{\partial x}>0, \dfrac{\partial \varphi(x,y,0)}{\partial y}>0$.

4. Numerical tests

In order to discuss the relationships among the computational stability of the conservative and nonconservative schemes of the two-dimensional nonlinear shallow water wave equations, the structure of the schemes, and form of the initial values, the following numerical experiments are performed. Two initial values are chosen as,

1. $u(x,y,0)=x$; $v(x,y,0)=y$; $\varphi(x,y,0)=g[1-\mathrm{e}^{-(x+y)}]$,

2. $u(x,y,0)=\sin 2\pi x$; $v(x,y,0)=\sin 2\pi y$; $\varphi(x,y,0)=g\cos 2\pi(x+y)$.

where $0\leqslant x\leqslant 1000$, $0\leqslant y\leqslant 1000$, $0\leqslant t\leqslant 10000$, and $h=20$.

Table 1. Computational results of numerical experiments

	scheme 1a	scheme 1b	scheme 2	scheme 3
initial value 1	stable	stable	stable	stable
initial value 2	stable	stable	unstable	unstable

Numerically, take $\Delta x = 1$, $\Delta t = 0.1$ for scheme 1a, 2, and 3. For Scheme 1a, ε is determined by Expression (4); for scheme 1b, ε is determined by Expression (4), and Δt by Expression (7). Here, scheme 1a is the explicit complete square conservative scheme whose time-step is constant, and scheme 1b the explicit complete square conservative scheme whose time-step is adjustable. The results are shown in Table 1.

From the results we can see that schemes 1a and 1b are stable. This shows that scheme 1a and 1b have a good square conservativeness. But scheme 1b with an adjustable time-step is more time-saving than scheme 1a with a constant time-step. Schemes 2 and 3 are stable for initial value 1, owing to the satisfying of Theorem 3. They are unstable for initial value 2, however, since the stability conditions of Theorem 3 are not satisfied. But in practical computations of atmosphere and ocean equations, the initial fields similar to initial value 1 do not exist. So Schemes 2 and 3 are not really suitable for the practical computation of atmosphere and ocean equations.

5. Conclusion

From the above discussions, some conclusions can be drawn. The computational stability of the conservative scheme is totally different from that of the nonconservative scheme. The computational stability of the conservative scheme is only concerned with the structure of the scheme. The computational stability of the nonconservative scheme depends not only on the structure of the scheme, but also on the form of the initial values and their partial derivatives.

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