Physical Mechanism and Model of Turbulent Cascades in a Barotropic Atmosphere

HUANG Feng* (黄 锋), and LIU Shikuo (刘式适)

School of Physics, Peking University, Beijing 100871

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ABSTRACT

In a barotropic atmosphere, new Reynolds mean momentum equations including turbulent viscosity, dispersion, and instability are used not only to derive the KdV-Burgers-Kuramoto equation but also to analyze the physical mechanism of the cascades of energy and enstrophy. It shows that it is the effects of dispersion and instability that result in the inverse cascade. Then based on the conservation laws of the energy and enstrophy, a cascade model is put forward and the processes of the cascades are described.

Key words: physical mechanism, cascade model, turbulence, barotropic atmosphere

1. Introduction

It is well known that the basic flow gains kinetic energy from eddies in atmospheric large-scale motion. This is an inverse energy cascade problem, which was called the negative viscosity phenomena by Starr (1966). In order to explain the phenomena, Liu and Liu (1992, 1995) reconsidered Prandtl's mixing length theory and obtained the modified Reynolds' averaged momentum equations that contain the turbulent viscosity and dispersion. It is stated that the turbulent dispersion effect is the main reason for the energy inversion described above, and the conditions of the inverse cascade are also obtained. At the same time, it has long been recognized (Fjørtoft, 1953) that the atmosphere on the largest scale behaves roughly like a two-dimensional incompressible fluid. Leith (1971) also considered that the barotropic models of the atmosphere, which have been moderately successful in describing the large-scale motions, are essentially two-dimensional and isotropic. For this two-dimensional turbulence, Kraichnan (1967) demonstrated that without considering viscosity and forcing, the energy and enstrophy were conserved and gave rise to two cascades, an inverse energy cascade and a direct enstrophy cascade, which were also got by Batchelor (1969) and Leith (1968). After that, this result was shown to be generic to two-dimensional turbulence with a generalized enstrophy (Shivamoggi, 2000). Previously, meteorologists paid more attention to the energy inversion, but no or less attention to the direct enstrophy cascade and the relations between them. So, in this paper, we will analyze the physical processes of the cascades and the relations between them. Then a model is put forward to give an analytical demonstration. At last, we will discuss the relations between Prandtl's mixing length theory and the cascade model.

2. KdV-Burgers-Kuramoto equation

Neglecting the atmospheric density change, the averaged equations of motion and the thermodynamic equation for an inviscid and adiabatic atmosphere can be written as

$$\frac{d\overline{u}}{dt} - f\overline{v} = -\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial x} - \left(\frac{\partial\overline{u'^2}}{\partial x} + \frac{\partial\overline{u'v'}}{\partial y} + \frac{\partial\overline{u'w'}}{\partial z}\right) , \quad (1a)$$

$$\frac{d\overline{v}}{dt} + f\overline{u} = -\frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial y} - \left(\frac{\partial\overline{v'u'}}{\partial x} + \frac{\partial\overline{v'^2}}{\partial y} + \frac{\partial\overline{v'w'}}{\partial z}\right) , \quad (1b)$$

$$\frac{d\overline{w}}{dt} = -g - \frac{1}{\overline{\rho}}\frac{\partial\overline{p}}{\partial z} - \left(\frac{\partial\overline{w'u'}}{\partial x} + \frac{\partial\overline{w'v'}}{\partial y} + \frac{\partial\overline{w'^2}}{\partial z}\right) , \quad (1c)$$

$$\frac{d\overline{\theta}}{dt} = -\left(\frac{\partial\overline{\theta'u'}}{\partial x} + \frac{\partial\overline{\theta'v'}}{\partial y} + \frac{\partial\overline{\theta'w'}}{\partial z}\right) . \quad (1d)$$

Here $\bar{p}$, $\bar{\rho}$, and $\bar{\theta}$ are average pressure, density, and potential temperature; $(\bar{u}, \bar{v}, \bar{w})$ is the averaged velocity, (u', v', w') is the fluctuating or disturbed velocity; θ' is the fluctuating or disturbed potential temperature;

*E-mail: hfenglw@163.com

g is the acceleration due to gravity; f is the Coriolis parameter; and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \overline{u}\frac{\partial}{\partial x} + \overline{v}\frac{\partial}{\partial y} + \overline{w}\frac{\partial}{\partial z} . \quad (2)$$

Applying Prandtl's mixing length theory to treat the second-order moments which are the average of the quadratic of fluctuating or disturbed quantities. Taking $-\overline{u'w'}$ as an example, we assume that there exists a mixing length l' in which the particle retains its properties and exchanges them with other particles after l'. Assuming $\bar{u}$ as $\bar{u}(z_0)$ and $\bar{u}(z)$ at $z = z_0$, $z = z$, respectively, then it produces a deviation u' when the particle at $z = z_0$ reaches $z = z$ and mixes with the neighboring air ($l' = z - z_1$). When only the first three terms are considered , $-\overline{u'w'}$ may be written as

$$-\overline{u'w'} = -\alpha\frac{\partial\overline{u}}{\partial z} - \beta\frac{\partial^2\overline{u}}{\partial z^2} - \gamma\frac{\partial^3\overline{u}}{\partial z^3} , \quad (3)$$

where

$$\alpha = -\overline{l'w'},\ \ \beta = -\frac{1}{2}\overline{l'^2w'},\ \ \gamma = -\frac{1}{6}\overline{l'^3w'} ,$$

$\alpha < 0$ is called the viscosity coefficient, β is called the dispersion coefficient, and $\gamma < 0$ is called the instability coefficient, so that

$$-\frac{\partial\overline{u'w'}}{\partial z} = -\alpha\frac{\partial^2\overline{u}}{\partial z^2} - \beta\frac{\partial^3\overline{u}}{\partial z^3} - \gamma\frac{\partial^4\overline{u}}{\partial z^4} . \quad (4)$$

Similarly, we have

$$-\frac{\partial\overline{\theta'w'}}{\partial z} = -\alpha'\frac{\partial^2\overline{\theta}}{\partial z^2} - \beta'\frac{\partial^3\overline{\theta}}{\partial z^3} - \gamma'\frac{\partial^4\overline{\theta}}{\partial z^4} , \quad (5)$$

where α', β', and γ' are named the coefficients of thermal diffusivity, thermal dispersion, and thermal convection, respectively.

If the difference in various directions is disregarded and the average symbols are neglected entirely, Eq. (1) then may be written as

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \alpha\nabla^2 u - \beta\nabla\cdot\Box u - \gamma\nabla^4 u , \quad (6a)$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \alpha\nabla^2 v - \beta\nabla\cdot\Box v - \gamma\nabla^4 v , \quad (6b)$$

$$\frac{dw}{dt} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} - \alpha\nabla^2 w - \beta\nabla\cdot\Box w - \gamma\nabla^4 w , \quad (6c)$$

$$\frac{d\theta}{dt} = -\alpha'\nabla^2\theta - \beta'\nabla\cdot\Box\theta - \gamma'\nabla^4\theta , \quad (6d)$$

where

$$\nabla \equiv \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z},$$

$$\Box \equiv \boldsymbol{i}\frac{\partial^2}{\partial x^2} + \boldsymbol{j}\frac{\partial^2}{\partial y^2} + \boldsymbol{k}\frac{\partial^2}{\partial z^2}.$$

We neglect the Coriolis force and the pressure gradient force, and only consider the x-direction. Thus Eq. (6a) can be rewritten as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} = 0 . \quad (7)$$

This is the KdV-Burgers-Kuramoto equation. Its linear form is as follows:

$$\frac{\partial u}{\partial t} + c_0\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} = 0 , \quad (8)$$

c_0 is a constant. Setting

$$u = \hat{u}e^{i(kx-\omega t)} , \quad (9)$$

where, k is wavenumber and ω is angular frequency. Substituting it into Eq. (8), thus gives

$$\omega = \left(kc_0 - \beta k^3\right) + i\left(\alpha k^2 - \gamma k^4\right) . \quad (10)$$

Therefore

$$u = \hat{u}e^{k^2\left(\alpha-\gamma k^2\right)t}\cdot e^{ik\left[x-\left(c_0-\beta k^2\right)t\right]} . \quad (11)$$

So, the phase velocity of the linear wave described by the KdV-Burgers-Kuramoto equation is $c = c_0 - \beta k^2$, and the group velocity is $c_{\rm g} = c_0 - 3\beta k^2$. Thus, β plays the role of the dispersion. $\beta\partial^2 u/\partial x^2$ is called the dispersion term.

From Eq. (11) we know that $\alpha > 0$ makes the amplitude of u increase with time, so it plays an unstable role. $\alpha < 0$ is the opposite case, which makes the amplitude of u decrease with time. Similarly, $\gamma > 0$ makes the amplitude of u decrease with time, so it plays a stable role. $\gamma < 0$ is on the contrary. Lorenz pointed out that the turbulent irregularity results from the instability of the fluid. That is to say, it has a sensitivity to the early conditions. So it is more suitable to propose the KdV-Burgers-Kuramoto equation including turbulent viscosity, dispersion, and stability effects for the modelling of turbulence.

3. Physical mechanism analysis of cascades

3.1 *The eddy kinetic energy and the mean energy*

From atmospheric dynamics, we know that in a barotropic atmosphere the transformation function between mean kinetic energy $K_{\rm m}$ and eddy kinetic energy $K_{\rm e}$ in the finite domain A is defined as

$$\{K_{\rm m}\ , K_{\rm e}\} = \iint\limits_A \left(-\overline{u'v'}\frac{\partial\overline{u}}{\partial y}\right)\delta A , \quad (12)$$

$\{K_{\rm m}, K_{\rm e}\} > 0$ implies that the mean kinetic energy is converted into the eddy kinetic energy, corresponding to the cascading process of turbulence. The case of $\{K_{\rm m}, K_{\rm e}\} < 0$ is the opposite, which implies that the eddy kinetic energy is converted into the mean kinetic energy, corresponding to the inversion energy cascade process.

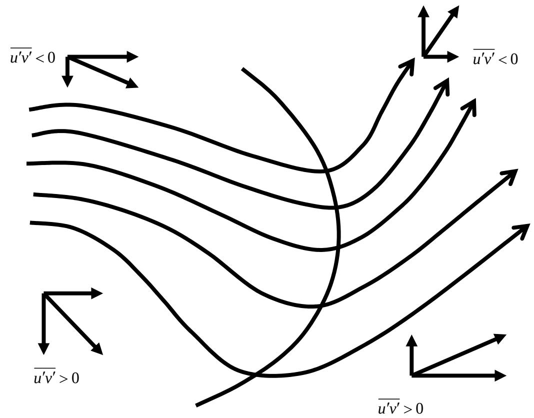

Fig. 1. Energy inversion of Rossby waves.

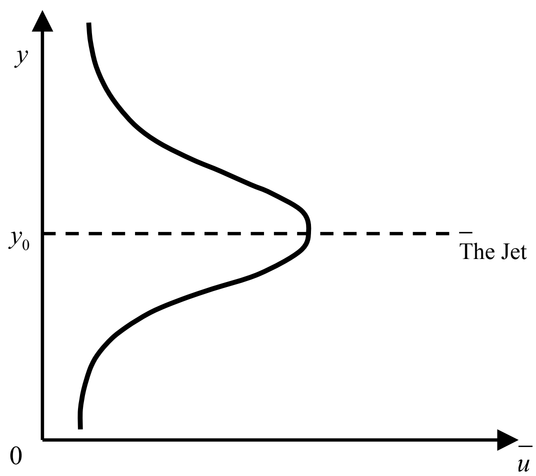

Fig. 2. The distribution of the jet.

Figure 1 is often seen in the isobaric surfaces. The maximum of $\bar{u}$ often appears in the middle of the trough-line. Obviously in the top of the figure, $\overline{u'v'} < 0$, which represents that west wind momentum is transformed to the south; in the bottom, $\overline{u'v'} > 0$, which represents that west wind momentum is transformed to the north. So the momentum is transformed from a smaller to a larger value of $\bar{u}$, the eddy kinetic energy is transformed to mean kinetic energy, and energy is concentrated in this process. This is just the energy inversion.

In the atmosphere, the distribution of the jet is like that shown in Fig. 2. On the north of the jet, $\partial\bar{u}/\partial y < 0$; on the south of the jet, $\partial\overline{u}/\partial y > 0$. Thus, we know that on the south of the jet $-\overline{u'v'}\partial\bar{u}/\partial y < 0$, that is to say $\{K_{\rm m}, K_{\rm e}\} < 0$, so in this area, the momentum is transformed from a smaller to a larger value of $\bar{u}$; similarly, we can analyze the north of the jet.

Energy inversion in Fig. 1 will also be analyzed in terms of Prandtl's mixing length theory. By using this theory, we will get

$$-\overline{u'v'} = -\alpha\frac{\partial\overline{u}}{\partial y} - \beta\frac{\partial^2\overline{u}}{\partial y^2} - \gamma\frac{\partial^3\overline{u}}{\partial y^3}\ , \tag{13}$$

where

$$\alpha = -\overline{l'v'},\quad \beta = -\frac{1}{2}\overline{l'^2v'},\quad \gamma = -\frac{1}{6}\overline{l'^3v'},$$

$-\alpha\partial\bar{u}/\partial y$ represents the dissipation effect, $-\beta\partial^2\overline{u}/\partial y^2$ represents the dispersion effect, and $-\gamma\partial^3\overline{u}/\partial y^3$ represents the instability effect.

Substituting Eq. (13) into Eq. (12), then we can get:

$$\{K_{\rm m}, K_{\rm e}\} = \iint\limits_A \left(-\alpha\frac{\partial\overline{u}}{\partial y} - \beta\frac{\partial^2\overline{u}}{\partial y^2} - \gamma\frac{\partial^3\overline{u}}{\partial y^3}\right)\frac{\partial\overline{u}}{\partial y}\delta A\ . \tag{14}$$

So, the necessary condition of energy inversion which represents that the eddy kinetic energy is transformed into the mean kinetic energy is

$$\iint\limits_A \left[\alpha\left(\frac{\partial\overline{u}}{\partial y}\right)^2 + \beta\frac{\partial\overline{u}}{\partial y}\frac{\partial^2\overline{u}}{\partial y^2} + \gamma\frac{\partial\overline{u}}{\partial y}\frac{\partial^3\overline{u}}{\partial y^3}\right]\delta A > 0\ . \tag{15}$$

From Eq. (7), the two sides of this equation are both multiplied by u and integrated; we consider that when $x \to \pm\infty$, $u \to 0$, and $\partial u/\partial x$, $\partial^2u/\partial x^2$, $\partial^3u/\partial x^3$ are limited, thus

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{1}{2}u^2dx = \int_{-\infty}^{\infty}\left[\alpha\left(\frac{\partial u}{\partial x}\right)^2 + \beta\frac{\partial u}{\partial x}\frac{\partial^2u}{\partial x^2} + \gamma\frac{\partial u}{\partial x}\frac{\partial^3u}{\partial x^3}\right]dx\ . \tag{16}$$

$\int_{-\infty}^{\infty}\frac{1}{2}u^2dx$ is the mean kinetic energy. Then the right-hand side of Eq. (16) is just the transformation of the eddy kinetic energy to the mean kinetic energy, namely, $\{K_{\rm e}, K_{\rm m}\}$. Therefore, only when

$$\int_{-\infty}^{\infty}\left[\alpha\left(\frac{\partial u}{\partial x}\right)^2 + \beta\frac{\partial u}{\partial x}\frac{\partial^2u}{\partial x^2} + \gamma\frac{\partial u}{\partial x}\frac{\partial^3u}{\partial x^3}\right]dx > 0,$$

can the energy inversion take place.

So, if only considering dissipation ($\alpha < 0,\ \beta = \gamma = 0$), the mean kinetic energy is converted to the eddy kinetic energy. Then there only exists the direct energy cascade. Only by adding the dispersion term or the instability term or both, can the inverse cascade happen. So the mechanism of the energy inversion is perhaps the dispersion and instability effects.

3.2 *The eddy enstrophy and the mean enstrophy*

From atmospheric dynamics, the mean enstrophy $Q^*_{\rm m}$ and the eddy enstrophy $Q^*_{\rm e}$ in the finite domain A

can be defined as

$$Q_{\rm m}^* = \iint\limits_A \rho \frac{1}{2} \overline{q_0}^2 \delta A\,, \quad Q_{\rm e}^* = \iint\limits_A \rho \frac{1}{2} \overline{q_0'^2} \delta A\,, \qquad (17)$$

where

$$\overline{q_0} = -\frac{\partial \overline{u}}{\partial y} - \lambda_0^2 \overline{\psi}$$

is the basic flow potential vorticity, $\overline{\psi}$ is the mean stream function, $\overline{u} = -\partial\overline{\psi}/\partial y$, and

$$\lambda_0 = f_0/c_0 \;\; (c_0 = \sqrt{gH});\;\; q_0' = \nabla_h^2 \psi' - \lambda_0^2 \psi'$$

is the disturbed potential vorticity, ψ' is the disturbed stream function, $u' = -\partial\psi'/\partial y$, and $v' = \partial\psi'/\partial x$.

Without considering viscosity, adiabatically and in the quasi-geostrophic model, the enstrophy is conserved. Then

$$\frac{\partial Q_{\rm m}^*}{\partial t} = -\iint\limits_A \rho \overline{q_0} \frac{\partial}{\partial y} \overline{q_0' v'} \delta A\,, \qquad (18)$$

$$\frac{\partial Q_{\rm e}^*}{\partial t} = \iint\limits_A \rho \overline{q_0} \frac{\partial}{\partial y} \overline{q_0' v'} \delta A\,. \qquad (19)$$

So

$$\{Q_{\rm m}, Q_{\rm e}\} = \iint\limits_A \overline{q_0} \frac{\partial}{\partial y} \overline{q_0' v'} dA = \iint\limits_A -\overline{q_0' v'} \frac{\partial \overline{q_0}}{\partial y} dA\,. \qquad (20)$$

From synoptic meteorology, referring to the sign analysis of the u', v' in Fig. 1, we can get the distribution of vorticity (see Fig. 3). From Fig. 3, it is obvious that

$$\frac{\partial \overline{q_0}}{\partial y} > 0,\;\; -\overline{q_0' v'} > 0,\;\; \text{so}\, \{Q_{\rm m}, Q_{\rm e}\} > 0.$$

This shows that when the energy cascade is inverse, the enstrophy cascade is direct. And we can also get that the area of the enstrophy cascade is between the high and low air pressures.

In fact, since

$$\overline{q_0} = -\frac{\partial \overline{u}}{\partial y} - \lambda_0^2 \overline{\psi},$$

then

$$\frac{\partial \overline{q_0}}{\partial y} = -\frac{\partial^2 \overline{u}}{\partial y^2} + \lambda_0^2 \overline{u}\,. \qquad (21)$$

Using scale analysis, we know that whether the sign of Eq. (21) is positive or not is determined by the term of $-\partial^2\overline{u}/\partial y^2$, whose sign is as according to Fig. 4.

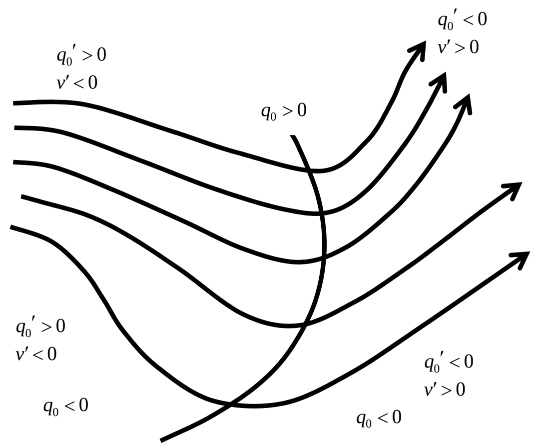

Fig. 3. The distribution of potential vorticity.

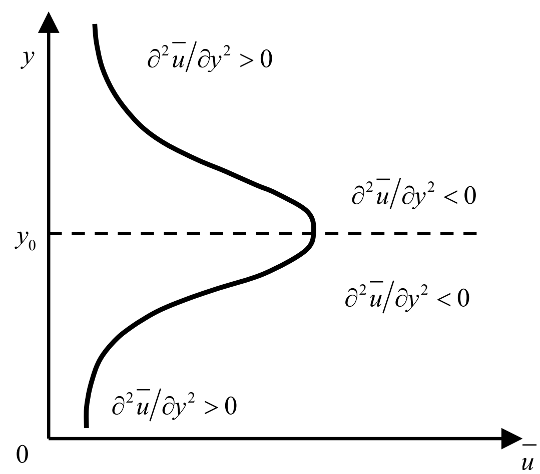

Fig. 4. The distribution of $-\partial^2\overline{u}/\partial y^2$.

Therefore, from south to north, on the south of the jet, the sign of $\partial\overline{q_0}/\partial y$ is from positive to negative; on the north of the jet, the sign is from negative to positive. From atmospheric dynamics,

$$-\overline{q_0' v'} = \frac{\partial}{\partial y} \overline{u' v'}.$$

Substituting it and Eq. (21) into Eq. (20), then

$$\{Q_{\rm m}, Q_{\rm e}\} = \iint\limits_A \frac{\partial}{\partial y} \overline{u' v'} \left(-\frac{\partial^2 \overline{u}}{\partial y^2} + \lambda_0^2 \overline{u} \right) dA\,. \qquad (22)$$

From Fig. 1, we can know the sign of $\partial\left(\overline{u'v'}\right)/\partial y$, so on the two sides of the jet, $\{Q_{\rm m}, Q_{\rm e}\} > 0$. This result is the same as that from Fig. 3.

In the above, we analyze the universal case and there exist two cascades. In the following, from the conversations laws, we will give an analytical demonstration of the inverse energy cascade and direct enstrophy cascade.

4. The conversation laws and the cascade's analytical demonstration

Referring to Liu and Liu (1991), in a barotropic atmosphere, the wave energy density $\mathcal{E}_r$ and the enstrophy density $\mathcal{F}_r$ of Rossby waves are given by

$$\mathcal{E}_r = \left(K_r^2 + \lambda_0^2\right) a_r^2/4 \ \ (r = 1, 2, ..., N) \ , \quad (23)$$

$$\mathcal{F}_r = \left(K_r^2 + \lambda_0^2\right)^2 a_r^2 \Big/ 2 \ \ (r = 1, 2, ..., N) \ , \quad (24)$$

where

$$\boldsymbol{K}_r = k_r \boldsymbol{i} + l_r \boldsymbol{j}$$

and

$$K_r = \sqrt{k_r^2 + l_r^2}.$$

k_r, l_r are wavenumbers of the rth Rossby wave in the x and y directions, respectively. a_r is the amplitude of the rth wave.

From atmospheric dynamics, in the quasi-geostrophic model, the energy, which contains the kinetic energy and the available potential energy, and enstrophy are conserved. These can be shown by Eqs. (23) and (24) in which the energy and enstrophy are measured by amplitudes. And we can also demonstrate that the net energy transfer must either be out of the middle wavenumber into both smaller and larger wavenumbers, or vice versa. It is so for the enstrophy also.

Fjørtoft (1953) found that only fractions of the initial energy can flow into smaller scales and that a greater fraction simultaneously has to flow to components with larger scales. Kraichnan (1967) demonstrated that it seemed to be more plausible to have net flow out of the middle wavenumber.

Based on the above discussions and from the work done by Shivamoggi (2000), we can consider a source in the spectral space at wavenumber k_0, whose total energy is E_0. This source would then decay via triadic interactions into two modes with wavenumbers $k_1 < k_0$ and $k_2 > k_0$, whose total energies are E_1 and E_2, respectively. Since energy and enstrophy are conserved during this decay, from Eqs. (23) and (24), we get

$$\left\{ \begin{array}{l} E_0 = E_1 + E_2 \\ k_0^2 E_0 = k_1^2 E_1 + k_2^2 E_2 \end{array} \right. \quad (25)$$

If greater fractions of the initial energy flow into larger wavenumbers, that is $E_1 \approx 0, E_2 \approx E_0$, then $k_2^2 E_2 > k_0^2 E_0$. The enstrophy will increase, which is not consistent with the enstrophy conservation. So, under the conditions of the conservations, the energy will flow into larger scales.

Setting

$$k_1^2 = \alpha_1 k_0^2, \ k_2^2 = \beta_1 k_0^2 \quad 0 < \alpha_1 < 1, \beta_1 > 1 \ , \quad (26)$$

$$E_1 = pE_0, \ E_2 = qE_0 \quad 0 < p < 1, 0 < q < 1 \ , \quad (27)$$

then

$$E_0 = \alpha_1 E_1 + \beta_1 E_2 = E_1 + E_2 \ . \quad (28)$$

So, from Eqs. (27) and (28)

$$(\alpha_1 - 1)\, pE_0 + (\beta_1 - 1)\,(1 - p)\, E_0 = 0 \ . \quad (29)$$

Thus

$$\alpha_1 = p, \qquad \beta_1 = 1 + p \ . \quad (30)$$

We can use lower and upper labels with brackets to represent the cascade steps and the modes after cascade, respectively. Then we know that the mode k_0 first decays into two modes:

$$k_{(1)}^{(0)} = p^{\frac{1-0}{2}} (1+p)^{\frac{0}{2}} k_0,$$

and

$$k_{(1)}^{(1)} = p^{\frac{1-1}{2}} (1+p)^{\frac{1}{2}} k_0,$$

with corresponding energies

$$E_{(1)}^{(0)} = \tbinom{1}{0} p^{1-0} (1-p)^0 E_0,$$

$$E_{(1)}^{(1)} = \tbinom{1}{1} p^{1-1} (1-p)^1 E_0,$$

and enstrophies

$$Q_{(1)}^{(0)} = \tbinom{1}{0} \left(p^2\right)^{1-0} \left(1-p^2\right)^0 Q_0,$$

$$Q_{(1)}^{(1)} = \tbinom{1}{1} \left(p^2\right)^{1-1} \left(1-p^2\right)^1 Q_0,$$

respectively.

In the second step of the cascade, the mode $k_{(1)}^{(0)}$ decays into a mode

$$k_{(2)}^{(0)} = p^{\frac{2-0}{2}} (1+p)^{\frac{0}{2}} k_0$$

and another mode

$$k_{(2)}^{(1)} = p^{\frac{2-1}{2}} (1+p)^{\frac{1}{2}} k_0,$$

while the mode $k_{(1)}^{(1)}$ decays into a mode

$$k_{(2)}^{(1)} = p^{\frac{2-1}{2}} (1+p)^{\frac{1}{2}} k_0$$

and another mode

$$k_{(2)}^{(2)} = p^{\frac{2-2}{2}} (1+p)^{\frac{2}{2}} k_0.$$

The energies of these three modes $k_{(2)}^{(0)}$, $k_{(2)}^{(1)}$, and $k_{(2)}^{(2)}$ are

$$E_{(2)}^{(0)} = \tbinom{2}{0} p^{2-0} (1-p)^0 E_0,$$

$$E_{(2)}^{(1)} = \tbinom{2}{1} p^{2-1} (1-p)^1 E_0,$$

$$E_{(2)}^{(2)} = \tbinom{2}{2} p^{2-2} (1-p)^2 E_0,$$

and the enstrophies

$$Q_{(2)}^{(0)} = \tbinom{2}{0} \left(p^2\right)^{2-0} \left(1-p^2\right)^0 Q_0,$$

$$Q_{(2)}^{(1)} = 2p^2 \left(1-p^2\right) Q_0 = \tbinom{2}{1} \left(p^2\right)^{2-1} \left(1-p^2\right)^1 Q_0,$$

and

$$Q_{(2)}^{(2)} = \binom{2}{2} \left(p^2\right)^{2-2} \left(1-p^2\right)^2 Q_0,$$

respectively.

Thus, at the nth step of the cascade, the wavenumber, energy, and enstrophy of each mode are correspondingly as follows:

$$k_{(n)}^{(r)} = p^{\frac{n-r}{2}} \left(1+p\right)^{\frac{r}{2}} k_0 \ , \tag{31}$$

$$E_{(n)}^{(r)} = \binom{n}{r} p^{n-r} \left(1-p\right)^r E_0 \ , \tag{32}$$

$$Q_{(n)}^{(r)} = \binom{n}{r} \left(p^2\right)^{n-r} \left(1-p^2\right)^r Q_0 \ . \tag{33}$$

Now, by the de Moivre-Laplace approximation, we have for the binomial distribution

$$\lim_{n\to\infty} \binom{n}{r} p^{n-r} \left(1-p\right)^r \approx \frac{1}{\sqrt{2\pi np\left(1-p\right)}} e^{-\frac{1}{2}\left(\frac{n-r-np}{\sqrt{np(1-p)}}\right)^2} , \tag{34}$$

so that the binomial distribution in Eq. (32) peaks at $r/n = 1-p$ as $n \to \infty$. This can be easily explained from the energy conservation. At the $(n-1)$th step,

$$E_{(n-1)}^{(r-1)} = \binom{n-1}{r-1} p^{n-r} \left(1-p\right)^{r-1} E_0$$

due to the energy conservation, and the maximum energy transformed from the $(n-1)$th step to the nth step is $E_{(n-1)}^{(r-1)}$, that is

$$E_{(n)}^{(r)} \Big/ E_{(n-1)}^{(r-1)} = \frac{1-p}{r/n} = 1;$$

and only when $r/n = 1-p$ as $n \to \infty$ can this take place. The corresponding wavenumber is given by

$$\lim_{n\to\infty} k_{(n)}^{(r)} = \lim_{n\to\infty} \left[p^p \left(1+p\right)^{1-p}\right]^{\frac{n}{2}} k_0 \ . \tag{35}$$

The binomial distribution in Eq. (33) peaks at $r/n = 1-p^2$ as $n \to \infty$. The corresponding wavenumber is given by

$$\lim_{n\to\infty} k_{(n)}^{(r)} = \lim_{n\to\infty} \left[p^{p^2} \left(1+p\right)^{1-p^2}\right]^{\frac{n}{2}} k_0 \ . \tag{36}$$

In order to evaluate the limit in Eqs. (35) and (36), it proves to be convenient to use the following result (Polya and Szego, 1978):

$$e^{\frac{p_1/a_1 \ln a_1 + p_2/a_2 \ln a_2}{p_1/a_1 + p_2/a_2}} < \frac{p_1+p_2}{p_1/a_1+p_2/a_2} \ , \tag{37}$$

$$\frac{p_1 a_1 + p_2 a_2}{p_1+p_2} < e^{\frac{p_1 a_1 \ln a_1 + p_2 a_2 \ln a_2}{p_1 a_1 + p_2 a_2}} \ , \tag{38}$$

where a_1, a_2 and p_1, p_2 are positive numbers so that $p_1 + p_2 = 1$ and $a_1 \neq a_2$.

Taking $p_1 = p^2$, $p_2 = 1-p^2$, $a_1 = p$, and $a_2 = 1+p$ we obtain from Eq. (37)

$$p^p \left(1+p\right)^{1-p} < 1, \quad 0 < p < 1 \ . \tag{39}$$

From Eq. (35), we have

$$\lim_{n\to\infty} \left[p^p \left(1+p\right)^{1-p}\right]^{\frac{n}{2}} k_0 \approx 0 \ . \tag{40}$$

Therefore, the peak of the energy distribution moves to the larger scales and the energy cascades inversely.

Similarly, taking $p_1 = p$, $p_2 = 1-p$, $a_1 = p$, and $a_2 = 1+p$, we obtain from the Eq. (38)

$$p^{p^2} \left(1+p\right)^{1-p^2} > 1 \ . \tag{41}$$

So, from Eq. (36)

$$\lim_{n\to\infty} \left[p^{p^2} \left(1+p\right)^{1-p^2}\right]^{\frac{n}{2}} k_0 \approx \infty \ . \tag{42}$$

Therefore, the peak of the enstrophy distribution moves to smaller scales. The enstrophy cascades directly. And we can also get the result that energy cascades inversely perhaps because the enstrophy is conserved or else it could not happen.

5. Discussions and conclusions

In fact, the cascade analytical demonstration does not account for the mechanism of the cascades of the energy and enstrophy. But from the expression of the $\{K_{\mathrm{m}}, K_{\mathrm{e}}\}$, we know that it is just the shear of $\bar{u}$ that causes the inverse cascade. Actually, when we obtain the KdV-Burgers-Kuramoto equation, it implies that the shear of the velocity field is the mechanism of the direct and inverse cascades. α, β, and γ show the effects of each term and determine the directions of the cascades.

Equation (1) also imply the conservations of the energy and enstrophy. That is, the equation that the conservations of the energy and enstrophy satisfy is the same as Eq. (1). The expressions differ just because one explains the phenomena from the side of dynamics, while the other from that of the conservations. Therefore, the KdV-Burgers-Kuramoto equation can successfully describe the turbulent motions as well as the processes of the cascades, and it can further be used to analyze the effects of each term in the processes. Using the conservation laws, we can show the directions of these processes. In a barotropic atmosphere, there exist the inverse energy cascade and the direct enstrophy cascade simultaneously.

Due to the complexities of the atmospheric motions, some problems need to be further discussed. In this paper, some phenomena have only been explained from the aspects of the physical mechanism and the mathematical model, and some plausible explainations have only been given. And the discussions are limited to a barotropic atmosphere; whether these results are generic or not needs to be further discussed.

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