The Relationship between Nonconservative Schemes and Initial Values of Nonlinear Evolution Equations

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ABSTRACT

For the nonconservative schemes of the nonlinear evolution equations, taking the one-dimensional shallow water wave equation as an example, the necessary conditions of computational stability are given. Based on numerical tests, the relationship between the nonlinear computational stability and the construction of difference schemes, as well as the form of initial values, is further discussed. It is proved through both theoretical analysis and numerical tests that if the construction of difference schemes is definite, the computational stability of nonconservative schemes is decided by the form of initial values.

Key words: nonlinear evolution equation, nonconservative scheme, initial value

1. Introduction

The movement, development and change of atmosphere and ocean in nature are nonlinear. The movement equations of the atmosphere and ocean belong to a kind of very complex nonlinear partial differential equation, which is often called the nonlinear evolution equation. Because of the complexity, it is often difficult to get an analytical solution. So to get a numerical solution from discrete forms of the equation is needed.

Energy conservation is one of the main characteristics for the short-range motion of the atmosphere and ocean. The finite-difference schemes are mostly employed to carry out the numerical solutions of this kind of problem, so it is key to know how to design long-time computationally stable difference schemes. To solve this problem, Zeng (1978), Zeng and Ji (1981) systematically studied the computational stability of the adiabatic or non-dissipative nonlinear evolution equations, discussed the reasons causing nonlinear computational instability, and constructed a computationally stable implicit complete square conservative difference scheme. Later, Ji and Wang (1991), Ji et al. (1998), and Wang and Ji (1990; 1994; 1995) constructed a computationally stable explicit complete square conservative difference scheme. Recently, Chen and Ji (2001) studied the energy-conserving semi-Lagrangian scheme. Ji et al. (2002) also discussed the relationship between total energy conservation and the symplectic algorithm. Lin et al. (2003) discussed the relationship of short-range motion of atmosphere and ocean between the conservative and non-conservative schemes, and further proved that it is stable to use the square conservative scheme for the numerical solution of the problem, while the non-conservative scheme is unstable. So the square conservative scheme for the solution of this kind of problem has more advantages.

Usually, the medium and long-range movements of atmosphere and ocean are of non-energy-conservation. Because of the nonlinear characteristics, the initial conditions are very important to the movements. So with the change of initial conditions, the medium and long-range movements of atmosphere and ocean will also change (Zeng, 1979). Finite-difference schemes are also employed to carry out the numerical solutions of the nonlinear atmospheric and oceanic equations. So it is very important to study the relationship between the difference scheme of the nonlinear evolution equation and the initial values. For the non-conservative schemes of the nonlinear evolution equation, Lin et al. (2000) gave a new method for judging the computational stability. Meanwhile, Lin et al. (2002) carried out a comparative analysis of the computational stability for conservative and non-conservative schemes of the nonlinear evolution equation, and proved that the computational stability of

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the conservative scheme is totally different from that of the non-conservative scheme. In this paper, taking the one-dimensional nonlinear shallow water wave equation as an example, the computational stability is analyzed for the different non-conservative schemes of the nonlinear evolution equation, and the relationship between the schemes and the initial values is discussed.

2. Equation and difference scheme

The one-dimensional nonlinear shallow water wave equations are:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \dfrac{\partial \varphi}{\partial x} = 0 , \\ \dfrac{\partial \varphi}{\partial t} + u\dfrac{\partial \varphi}{\partial x} + (\Phi + \varphi)\dfrac{\partial u}{\partial x} = 0 , \\ u(x,0) = u_0(x) , \\ \varphi(x,0) = \varphi_0(x) . \end{cases} \tag{1}$$

where $\varphi = g\xi$, $\Phi = gh$, ξ is surface elevation, g is gravitational acceleration, h is water depth, and g and h are constants.

For equation (1), we take C grids and use the following difference schemes:

Scheme 1 (CTCS scheme)

$$\frac{u_{j+1/2}^{n+1} - u_{j+1/2}^{n-1}}{2\Delta t} + u_{j+1/2}^{n}\frac{u_{j+3/2}^{n} - u_{j-1/2}^{n}}{2\Delta x} + \frac{\varphi_{j+1}^{n} - \varphi_{j-1}^{n}}{2\Delta x} = 0 , \tag{2}$$

$$\frac{\varphi_{j}^{n+1} - \varphi_{j}^{n-1}}{2\Delta t} + u_{j+1/2}^{n}\frac{\varphi_{j+1}^{n} - \varphi_{j-1}^{n}}{2\Delta x} + (\Phi + \varphi_j^n)\frac{u_{j+3/2}^{n} - u_{j-1/2}^{n}}{2\Delta x} = 0 . \tag{3}$$

Scheme 2 (FTBS scheme)

$$\frac{u_{j+1/2}^{n+1} - u_{j+1/2}^{n}}{\Delta t} + u_{j+1/2}^{n}\frac{u_{j+1/2}^{n} - u_{j-1/2}^{n}}{\Delta x} + \frac{\varphi_{j}^{n} - \varphi_{j-1}^{n}}{\Delta x} = 0 , \tag{4}$$

$$\frac{\varphi_{j}^{n+1} - \varphi_{j}^{n}}{\Delta t} + u_{j+1/2}^{n}\frac{\varphi_{j}^{n} - \varphi_{j-1}^{n}}{\Delta x} + (\Phi + \varphi_j^n)\frac{u_{j+1/2}^{n} - u_{j-1/2}^{n}}{\Delta x} = 0 . \tag{5}$$

Scheme 3 (BTCS scheme)

$$\frac{u_{j+1/2}^{n+1} - u_{j+1/2}^{n}}{\Delta t} + u_{j+1/2}^{n}\frac{u_{j+3/2}^{n+1} - u_{j-1/2}^{n+1}}{2\Delta x} + \frac{\varphi_{j+1}^{n+1} - \varphi_{j-1}^{n+1}}{2\Delta x} = 0 , \tag{6}$$

$$\frac{\varphi_{j}^{n+1} - \varphi_{j}^{n}}{\Delta t} + u_{j+1/2}^{n}\frac{\varphi_{j+1}^{n+1} - \varphi_{j-1}^{n+1}}{2\Delta x} + (\Phi + \varphi_j^n)\frac{u_{j+3/2}^{n+1} - u_{j-1/2}^{n+1}}{2\Delta x} = 0 . \tag{7}$$

Scheme 4 (Lax-Wendroff scheme)

$$\frac{u_{j+1/2}^{n+1} - u_{j+1/2}^{n}}{\Delta t} + u_{j+1/2}^{n}\frac{u_{j+3/2}^{n} - u_{j-1/2}^{n}}{2\Delta x} + \frac{\varphi_{j+1}^{n} - \varphi_{j-1}^{n}}{2\Delta x} = \frac{[(u_{j+1/2}^{n})^2 + \Phi + \varphi_j^n]\Delta t}{2\Delta x^2}(u_{j+3/2}^{n} - 2u_{j+1/2}^{n} + u_{j-1/2}^{n}) + \frac{u_{j+1/2}^{n}\Delta t}{\Delta x^2}(\varphi_{j+1}^{n} - 2\varphi_j^n + \varphi_{j-1}^{n}) , \tag{8}$$

$$\frac{\varphi_{j}^{n+1} - \varphi_{j}^{n}}{\Delta t} + u_{j+1/2}^{n}\frac{\varphi_{j+1}^{n} - \varphi_{j-1}^{n}}{2\Delta x} + (\Phi + \varphi_j^n)\frac{u_{j+3/2}^{n} - u_{j-1/2}^{n}}{2\Delta x} = \frac{[(u_{j+1/2}^{n})^2 + \Phi + \varphi_j^n]\Delta t}{2\Delta x^2}(\varphi_{j+1}^{n} - 2\varphi_j^n + \varphi_{j-1}^{n}) + \frac{u_{j+1/2}^{n}(\Phi + \varphi_j^n)\Delta t}{\Delta x^2}(u_{j+3/2}^{n} - 2u_{j+1/2}^{n} + u_{j-1/2}^{n}) . \tag{9}$$

3. Computational stability analysis of the difference schemes

According to Lin et al. (2000), the computational stability of Schemes 1–4 can be analyzed. Scheme 1 is taken as the example. By means of Taylor expansion for (2) and (3), omitting superscripts and subscripts, we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial x} = -\frac{1}{6}\frac{\partial^3 u}{\partial t^3}\Delta t^2 - \frac{1}{6}u\frac{\partial^3 u}{\partial x^3}\Delta x^2 - \frac{1}{6}\frac{\partial^3 \varphi}{\partial x^3}\Delta x^2 + O(\Delta t^4, \Delta x^4) , \tag{10}$$

$$\frac{\partial \varphi}{\partial t} + u\frac{\partial \varphi}{\partial x} + (\Phi + \varphi)\frac{\partial u}{\partial x} = -\frac{1}{6}\frac{\partial^3 \varphi}{\partial t^3}\Delta t^2 - \frac{1}{6}u\frac{\partial^3 \varphi}{\partial x^3}\Delta x^2 - \frac{1}{6}(\Phi + \varphi)\frac{\partial^3 u}{\partial x^3}\Delta x^2 + O(\Delta t^4, \Delta x^4) . \tag{11}$$

From (10) and (11), we can see

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - \frac{\partial \varphi}{\partial x} + O(\Delta t^2, \Delta x^2) . \tag{12}$$

$$\frac{\partial \varphi}{\partial t} = -u\frac{\partial \varphi}{\partial x} - (\Phi + \varphi)\frac{\partial u}{\partial x} + O(\Delta t^2, \Delta x^2) . \tag{13}$$

Taking the derivative with respect to t in (12), we have

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\partial u}{\partial t}\frac{\partial u}{\partial x} - u\frac{\partial^2 u}{\partial t \partial x} - \frac{\partial^2 \varphi}{\partial t \partial x} + O(\Delta t^2, \Delta x^2) \,. \tag{14}$$

Taking the derivative with respect to x in (12), we have

$$\frac{\partial^2 u}{\partial t \partial x} = -\left(\frac{\partial u}{\partial x}\right)^2 - u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 \varphi}{\partial x^2} + O(\Delta t^2, \Delta x^2) \,. \tag{15}$$

Taking the derivative with respect to x in (13), we have

$$\begin{aligned}\frac{\partial^2 \varphi}{\partial t \partial x} &= -2\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x} - u\frac{\partial^2 \varphi}{\partial x^2} \\ &- (\Phi + \varphi)\frac{\partial^2 u}{\partial x^2} + O(\Delta t^2, \Delta x^2) \,. \end{aligned}\tag{16}$$

Substituting (12), (15), and (16) into (14), we have

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= 2u\left(\frac{\partial u}{\partial x}\right)^2 + 3\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x} + u^2\frac{\partial^2 u}{\partial x^2} \\ &+ 2u\frac{\partial^2 \varphi}{\partial x^2} + (\Phi + \varphi)\frac{\partial^2 u}{\partial x^2} + O(\Delta t^2, \Delta x^2) \,. \end{aligned}\tag{17}$$

Similarly, we can obtain

$$\begin{aligned}\frac{\partial^2 \varphi}{\partial t^2} &= \left(\frac{\partial \varphi}{\partial x}\right)^2 + 4u\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x} + 2(\Phi+\varphi)\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 \varphi}{\partial x^2} \\ &+ (\Phi+\varphi)\frac{\partial^2 \varphi}{\partial x^2} + 2(\Phi+\varphi)u\frac{\partial^2 u}{\partial x^2} + O(\Delta t^2, \Delta x^2) \,, \end{aligned}\tag{18}$$

$$\begin{aligned}\frac{\partial^3 u}{\partial t^3} &= -6u\left(\frac{\partial u}{\partial x}\right)^3 - 11\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial \varphi}{\partial x} - 9u^2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} \\ &- 12u\frac{\partial \varphi}{\partial x}\frac{\partial^2 u}{\partial x^2} - 15u\frac{\partial u}{\partial x}\frac{\partial^2 \varphi}{\partial x^2} - 5\frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x^2} \\ &- 7(\Phi + \varphi)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - u^3\frac{\partial^3 u}{\partial x^3} - 3(\Phi + \varphi)u\frac{\partial^3 u}{\partial x^3} \\ &- 3u^2\frac{\partial^3 \varphi}{\partial x^3} - (\Phi + \varphi)\frac{\partial^3 \varphi}{\partial x^3} + O(\Delta t^2, \Delta x^2) \,, \end{aligned}\tag{19}$$

$$\begin{aligned}\frac{\partial^3 \varphi}{\partial t^3} &= -18u\frac{\partial \varphi}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2 - 8\frac{\partial u}{\partial x}\left(\frac{\partial \varphi}{\partial x}\right)^2 \\ &- 6(\Phi + \varphi)\left(\frac{\partial u}{\partial x}\right)^3 - 9u^2\frac{\partial \varphi}{\partial x}\frac{\partial^2 u}{\partial x^2} - 9u^2\frac{\partial u}{\partial x}\frac{\partial^2 \varphi}{\partial x^2} \\ &- 9u\frac{\partial \varphi}{\partial x}\frac{\partial^2 \varphi}{\partial x^2} - 8(\Phi + \varphi)\frac{\partial u}{\partial x}\frac{\partial^2 \varphi}{\partial x^2} - 18(\Phi + \varphi)u\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} \\ &- 7(\Phi + \varphi)\frac{\partial \varphi}{\partial x}\frac{\partial^2 u}{\partial x^2} - u^3\frac{\partial^3 \varphi}{\partial x^3} - 3(\Phi + \varphi)u\frac{\partial^3 \varphi}{\partial x^3} \\ &- 3(\Phi + \varphi)u^2\frac{\partial^3 u}{\partial x^3} - (\Phi + \varphi)^2\frac{\partial^3 u}{\partial x^3} + O(\Delta t^2, \Delta x^2) \,. \end{aligned}\tag{20}$$

Substituting (19) and (20) into (10) and (11), we obtain the modified partial differential equation of Scheme 1:

$$\begin{aligned}\frac{\partial u}{\partial t} &+ u\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial x} = u\left(\frac{\partial u}{\partial x}\right)^3\Delta t^2 + \frac{11}{6}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial \varphi}{\partial x}\Delta t^2 \\ &+ \frac{1}{6}\Delta t^2\left[9u^2\frac{\partial u}{\partial x} + 12u\frac{\partial \varphi}{\partial x} + 7(\Phi + \varphi)\frac{\partial u}{\partial x}\right]\frac{\partial^2 u}{\partial x^2} \\ &+ \frac{1}{6}\Delta t^2\left(15u\frac{\partial u}{\partial x} + 5\frac{\partial \varphi}{\partial x}\right)\frac{\partial^2 \varphi}{\partial x^2} + \frac{1}{6}u^3\Delta t^2\frac{\partial^3 u}{\partial x^3} \\ &+ \frac{1}{2}(\Phi + \varphi)u\Delta t^2\frac{\partial^3 u}{\partial x^3} + \frac{1}{2}u^2\Delta t^2\frac{\partial^3 \varphi}{\partial x^3} \\ &+ \frac{1}{6}(\Phi + \varphi)\Delta t^2\frac{\partial^3 \varphi}{\partial x^3} - \frac{1}{6}u\Delta x^2\frac{\partial^3 u}{\partial x^3} \\ &- \frac{1}{6}\Delta x^2\frac{\partial^3 \varphi}{\partial x^3} + O(\Delta t^4, \Delta t^2\Delta x^2) \,. \end{aligned}\tag{21}$$

$$\begin{aligned}\frac{\partial \varphi}{\partial t} &+ u\frac{\partial \varphi}{\partial x} + (\Phi + \varphi)\frac{\partial u}{\partial x} = 3u\frac{\partial \varphi}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2\Delta t^2 \\ &+ \frac{4}{3}\frac{\partial u}{\partial x}\left(\frac{\partial \varphi}{\partial x}\right)^2\Delta t^2 + (\Phi + \varphi)\left(\frac{\partial u}{\partial x}\right)^3\Delta t^2 \\ &+ \frac{1}{6}\Delta t^2\left[9u^2\frac{\partial \varphi}{\partial x} + 18(\Phi + \varphi)u\frac{\partial u}{\partial x} + 7(\Phi + \varphi)\frac{\partial \varphi}{\partial x}\right]\frac{\partial^2 u}{\partial x^2} \\ &+ \frac{1}{6}\Delta t^2\left[9u^2\frac{\partial u}{\partial x} + 9u\frac{\partial \varphi}{\partial x} + 8(\Phi + \varphi)\frac{\partial u}{\partial x}\right]\frac{\partial^2 \varphi}{\partial x^2} \\ &+ \frac{1}{6}u^3\Delta t^2\frac{\partial^3 \varphi}{\partial x^3} + \frac{1}{2}(\Phi + \varphi)u\Delta t^2\frac{\partial^3 \varphi}{\partial x^3} \\ &+ \frac{1}{2}(\Phi + \varphi)u^2\Delta t^2\frac{\partial^3 u}{\partial x^3} + \frac{1}{6}(\Phi + \varphi)^2\Delta t^2\frac{\partial^3 u}{\partial x^3} \\ &- \frac{1}{6}u\Delta x^2\frac{\partial^3 \varphi}{\partial x^3} - \frac{1}{6}(\Phi + \varphi)\Delta x^2\frac{\partial^3 u}{\partial x^3} \\ &+ O(\Delta t^4, \Delta t^2\Delta x^2) \,. \end{aligned}\tag{22}$$

Similarly, we can obtain the modified partial differential equations of Schemes 2–4:

$$\begin{aligned}\frac{\partial u}{\partial t} &+ u\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial x} = -u\left(\frac{\partial u}{\partial x}\right)^2\Delta t - \frac{3}{2}\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x}\Delta t \\ &+ u\left(\frac{\partial u}{\partial x}\right)^3\Delta t^2 + \frac{11}{6}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial \varphi}{\partial x}\Delta t^2 \\ &+ \frac{1}{2}\left[u\Delta x - (u^2 + \Phi + \varphi)\Delta t\right]\frac{\partial^2 u}{\partial x^2} \\ &+ \frac{1}{2}(\Delta x - 2u\Delta t)\frac{\partial^2 \varphi}{\partial x^2} \\ &+ \frac{1}{6}\Delta t^2\left[9u^2\frac{\partial u}{\partial x} + 12u\frac{\partial \varphi}{\partial x} + 7(\Phi + \varphi)\frac{\partial u}{\partial x}\right]\frac{\partial^2 u}{\partial x^2}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{6}\Delta t^2\left(15u\frac{\partial u}{\partial x}+5\frac{\partial\varphi}{\partial x}\right)\frac{\partial^2\varphi}{\partial x^2}+\frac{1}{6}u^3\Delta t^2\frac{\partial^3 u}{\partial x^3}\\
&+\frac{1}{2}(\Phi+\varphi)u\Delta t^{22}\frac{\partial^3 u}{\partial x^3}+\frac{1}{2}u^2\Delta t^2\frac{\partial^3\varphi}{\partial x^3}\\
&+\frac{1}{6}(\Phi+\varphi)\Delta t^2\frac{\partial^3\varphi}{\partial x^3}-\frac{1}{6}u\Delta x^2\frac{\partial^3 u}{\partial x^3}\\
&-\frac{1}{6}\Delta x^2\frac{\partial^3\varphi}{\partial x^3}+O(\Delta t^2,\Delta t\Delta x)\,, \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial t}+u\frac{\partial\varphi}{\partial x}+(\Phi+\varphi)\frac{\partial u}{\partial x}=-2u\frac{\partial u}{\partial x}\frac{\partial\varphi}{\partial x}\Delta t\\
&-\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2\Delta t-(\Phi+\varphi)\left(\frac{\partial u}{\partial x}\right)^2\Delta t\\
&+3u\frac{\partial\varphi}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2\Delta t^2+\frac{4}{3}\frac{\partial u}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)^2\Delta t^2\\
&+(\Phi+\varphi)\left(\frac{\partial u}{\partial x}\right)^3\Delta t^2+\frac{1}{2}(\Phi+\varphi)(\Delta x-2u\Delta t)\frac{\partial^2 u}{\partial x^2}\\
&+\frac{1}{2}\left[u\Delta x-(u^2+\Phi+\varphi)\Delta t\right]\frac{\partial^2\varphi}{\partial x^2}\\
&+\frac{1}{6}\Delta t^2\left[9u^2\frac{\partial\varphi}{\partial x}+18(\Phi+\varphi)u\frac{\partial u}{\partial x}+7(\Phi+\varphi)\frac{\partial\varphi}{\partial x}\right]\frac{\partial^2 u}{\partial x^2}\\
&+\frac{1}{6}\Delta t^2\left[9u^2\frac{\partial u}{\partial x}+9u\frac{\partial\varphi}{\partial x}+8(\Phi+\varphi)\frac{\partial u}{\partial x}\right]\frac{\partial^2\varphi}{\partial x^2}\\
&+\frac{1}{6}u^3\Delta t^2\frac{\partial^3\varphi}{\partial x^3}+\frac{1}{2}(\Phi+\varphi)u\Delta t^2\frac{\partial^3\varphi}{\partial x^3}\\
&+\frac{1}{2}(\Phi+\varphi)u^2\Delta t^2\frac{\partial^3 u}{\partial x^3}+\frac{1}{6}(\Phi+\varphi)^2\Delta t^2\frac{\partial^3 u}{\partial x^3}\\
&-\frac{1}{6}u\Delta x^{22}\frac{\partial^3\varphi}{\partial x^3}-\frac{1}{6}(\Phi+\varphi)\Delta x^2\frac{\partial^3 u}{\partial x^3}\\
&+O(\Delta t^2,\Delta t\Delta x)\,, \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial\varphi}{\partial x}=\frac{1}{2}\frac{\partial u}{\partial x}\frac{\partial\varphi}{\partial x}\Delta t-\frac{7}{6}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial\varphi}{\partial x}\Delta t^2\\
&+\frac{1}{2}(u^2+\Phi+\varphi)\Delta t\frac{\partial^2 u}{\partial x^2}+u\Delta t\frac{\partial^2\varphi}{\partial x^2}\\
&-\frac{1}{6}\Delta t^2\left[9u^2\frac{\partial u}{\partial x}+18u\frac{\partial\varphi}{\partial x}+11(\Phi+\varphi)\frac{\partial u}{\partial x}\right]\frac{\partial^2 u}{\partial x^2}\\
&-\frac{1}{6}\Delta t^2\left(18u\frac{\partial u}{\partial x}+4\frac{\partial\varphi}{\partial x}\right)\frac{\partial^2\varphi}{\partial x^2}-\frac{1}{3}u^3\Delta t^2\frac{\partial^3 u}{\partial x^3}\\
&-(\Phi+\varphi)u\Delta t^2\frac{\partial^3 u}{\partial x^3}-u^2\Delta t^2\frac{\partial^3\varphi}{\partial x^3}\\
&-\frac{1}{3}(\Phi+\varphi)\Delta t^2\frac{\partial^3\varphi}{\partial x^3}-\frac{1}{6}u\Delta x^2\frac{\partial^3 u}{\partial x^3}\\
&-\frac{1}{6}\Delta x^2\frac{\partial^3\varphi}{\partial x^3}+O(\Delta t^2,\Delta t\Delta x^2)\,, \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial t}+u\frac{\partial\varphi}{\partial x}+(\Phi+\varphi)\frac{\partial u}{\partial x}=-\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2\Delta t\\
&+\frac{4}{3}\frac{\partial u}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)^2\Delta t^2+(\Phi+\varphi)u\Delta t\frac{\partial^2 u}{\partial x^2}\\
&+\frac{1}{2}(u^2+\Phi+\varphi)\Delta t\frac{\partial^2\varphi}{\partial x^2}\\
&-\frac{1}{6}\Delta t^2\left[9u^2\frac{\partial\varphi}{\partial x}+18(\Phi+\varphi)u\frac{\partial u}{\partial x}+5(\Phi+\varphi)\frac{\partial\varphi}{\partial x}\right]\frac{\partial^2 u}{\partial x^2}\\
&-\frac{1}{6}\Delta t^2\left[9u^2\frac{\partial u}{\partial x}+7(\Phi+\varphi)\frac{\partial u}{\partial x}\right]\frac{\partial^2\varphi}{\partial x^2}\\
&-\frac{1}{3}u^3\Delta t^2\frac{\partial^3\varphi}{\partial x^3}-(\Phi+\varphi)u\Delta t^2\frac{\partial^3\varphi}{\partial x^3}\\
&-(\Phi+\varphi)u^2\Delta t^2\frac{\partial^3 u}{\partial x^3}-\frac{1}{3}(\Phi+\varphi)^2\Delta t^2\frac{\partial^3 u}{\partial x^3}\\
&-\frac{1}{6}u\Delta x^{22}\frac{\partial^3\varphi}{\partial x^3}-\frac{1}{6}(\Phi+\varphi)\Delta x^2\frac{\partial^3 u}{\partial x^3}\\
&+O(\Delta t^2,\Delta t\Delta x^2)\,, \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial\varphi}{\partial x}=-u\left(\frac{\partial u}{\partial x}\right)^2\Delta t\\
&-\frac{3}{2}\frac{\partial u}{\partial x}\frac{\partial\varphi}{\partial x}\Delta t+u\left(\frac{\partial u}{\partial x}\right)^3\Delta t^2+\frac{11}{6}\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial\varphi}{\partial x}\Delta t^2\\
&+\frac{1}{6}\Delta t^2\left[9u^2\frac{\partial u}{\partial x}+12u\frac{\partial\varphi}{\partial x}+7(\Phi+\varphi)\frac{\partial u}{\partial x}\right]\frac{\partial^2 u}{\partial x^2}\\
&+\frac{1}{6}\Delta t^2\left(15u\frac{\partial u}{\partial x}+5\frac{\partial\varphi}{\partial x}\right)\frac{\partial^2\varphi}{\partial x^2}+\frac{1}{6}u^3\Delta t^2\frac{\partial^3 u}{\partial x^3}\\
&+\frac{1}{2}(\Phi+\varphi)u^2\frac{\partial^3 u}{\partial x^3}+\frac{1}{2}u^2\Delta t^2\frac{\partial^3\varphi}{\partial x^3}\\
&+\frac{1}{6}(\Phi+\varphi)\Delta t^2\frac{\partial^3\varphi}{\partial x^3}-\frac{1}{6}u\Delta x^2\frac{\partial^3 u}{\partial x^3}\\
&-\frac{1}{6}\Delta x^2\frac{\partial^3\varphi}{\partial x^3}+O(\Delta t^2,\Delta t\Delta x^2)\,. \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial\varphi}{\partial t}+u\frac{\partial\varphi}{\partial x}+(\Phi+\varphi)\frac{\partial u}{\partial x}=-2u\frac{\partial u}{\partial x}\frac{\partial\varphi}{\partial x}\Delta t\\
&-\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2\Delta t-(\Phi+\varphi)\left(\frac{\partial u}{\partial x}\right)^2\Delta t
\end{aligned}
$$

$$+ 3u\frac{\partial \varphi}{\partial x}\left(\frac{\partial u}{\partial x}\right)^2 \Delta t^2 + \frac{4}{3}\frac{\partial u}{\partial x}\left(\frac{\partial \varphi}{\partial x}\right)^2 \Delta t^2$$

$$+ (\Phi+\varphi)\left(\frac{\partial u}{\partial x}\right)^3 \Delta t^2$$

$$+ \frac{1}{6}\Delta t^2\left[9u^2\frac{\partial \varphi}{\partial x} + 18(\Phi+\varphi)u\frac{\partial u}{\partial x} + 7(\Phi+\varphi)\frac{\partial \varphi}{\partial x}\right]\frac{\partial^2 u}{\partial x^2}$$

$$+ \frac{1}{6}\Delta t^2\left[9u^2\frac{\partial u}{\partial x} + 9u\frac{\partial \varphi}{\partial x} + 8(\Phi+\varphi)\frac{\partial u}{\partial x}\right]\frac{\partial^2 \varphi}{\partial x^2}$$

$$+ \frac{1}{6}u^3\Delta t^2\frac{\partial^3 \varphi}{\partial x^3} + \frac{1}{2}(\Phi+\varphi)u\Delta t^2\frac{\partial^3 \varphi}{\partial x^3}$$

$$+ \frac{1}{2}(\Phi+\varphi)u^2\Delta t^2\frac{\partial^3 u}{\partial x^3} + \frac{1}{6}(\Phi+\varphi)^2\Delta t^2\frac{\partial^3 u}{\partial x^3}$$

$$- \frac{1}{6}u\Delta x^{22}\frac{\partial^3 \varphi}{\partial x^3} - \frac{1}{6}(\Phi+\varphi)\Delta x^2\frac{\partial^3 u}{\partial x^3}$$

$$+ O(\Delta t^2, \Delta t\Delta x^2)\,. \tag{28}$$

The Schemes 1–4 are stable only if the second-order dissipative coefficients on the right of (21)–(28) are positive (Wu and Han, 1988). For certain, they must be positive when $t = 0$ (Lin et al., 2000). Hence, we have the following theorems.

Theorem 1. For the Scheme 1 (CTCS scheme) and Scheme 4 (Lax-Wendroff scheme) of the one-dimensional nonlinear shallow water wave equation, the necessary conditions of computational stability are

(1) $u(x,0) > 0$;

(2) $\dfrac{\partial u(x,0)}{\partial x} \geqslant 0$;

(3) $\dfrac{\partial \varphi(x,0)}{\partial x} > 0$.

Theorem 2. For Scheme 2 (FTBS scheme) of the one-dimensional nonlinear shallow water wave equation, the necessary conditions of computational stability are

(1) $u(x,0) > 0$;

(2) $\dfrac{\partial u(x,0)}{\partial x} \geqslant 0$;

(3) $\dfrac{\partial \varphi(x,0)}{\partial x} > 0$;

(4) $\dfrac{u(x,0)\Delta t}{\Delta x} \leqslant \min\{\dfrac{1}{2}\,, \dfrac{u^2(x,0)}{u^2(x,0)+\Phi+\varphi(x,0)}\}$.

Theorem 3. For Scheme 3 (BTCS scheme) of the one-dimensional nonlinear shallow water wave equation, the necessary conditions of computational stability are

(1) $u(x,0) > 0$;

(2) $\dfrac{\partial u(x,0)}{\partial x} \leqslant 0$;

(3) $\dfrac{\partial \varphi(x,0)}{\partial x} \leqslant 0$.

4. Numerical tests

In order to verify the relationship of the computational stability between the difference schemes of the nonlinear evolution equations and the initial values, for Schemes 1–4, we perform the following numerical experiments. Two initial values are chosen,

(1) $u(x,0) = x$; $\varphi(x,0) = g\left(1 - \mathrm{e}^{-x}\right)$,

(2) $u(x,0) = \sin 2\pi x$; $\varphi(x,0) = g\cos 2\pi x$,

where $0 \leqslant x \leqslant 10$, $0 \leqslant t \leqslant 100$, and $h = 10$.

Numerically, we take $\Delta x = 0.1$ and $\Delta t = 0.01$. The results are shown in Table 1.

From the results we can see that Schemes 1, 2, and 4 are stable for initial value 1 because Theorems 1 and 2 are satisfied. Schemes 1, 2, and 4 are unstable for initial value 2, owing to the violation of Theorems 1 and 2. Scheme 3 is unstable for initial values 1 and 2, however, since the stability conditions of Theorem 3 are not satisfied.

5. Conclusion

From the discussions above, some conclusions can be drawn.

(1) There is a very close relationship for the computational stability between the non-conservative scheme of the nonlinear evolution equation and the initial values. If the construction of the difference schemes is definite, the computational stability of the non-conservative schemes is decided by the form of the initial values. It is proved that the movement of the discrete nonlinear evolution equation is still based on the initial field.

Table 1. Computational results of numerical experiments.

	Scheme 1	Scheme 2	Scheme 3	Scheme 4
initial value 1	stable	stable	unstable	stable
initial value 2	unstable	unstable	unstable	unstable

(2) The stability criteria from the computational stability analysis are really the necessary conditions of computational stability of the non-conservative schemes of the nonlinear evolution equation.

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