The Structure and Bifurcation of Atmospheric Motions

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(Received 12 June 2003; revised 26 December 2003)

ABSTRACT

The 3-D spiral structure resulting from the balance between the pressure gradient force, Coriolis force, and viscous force is a common atmospheric motion pattern. If the nonlinear advective terms are considered, this typical pattern can be bifurcated. It is shown that the surface low pressure with convergent cyclonic vorticity and surface high pressure with divergent anticyclonic vorticity are all stable under certain conditions. The anomalous structure with convergent anticyclonic vorticity is always unstable. But the anomalous weak high pressure structure with convergent cyclonic vorticity can exist, and this denotes the cyclone's dying out.

Key words: horizontal divergence, vertical vorticity, spiral structure, bifurcation

1. Introduction

Liu et al. (2000, 2003) discussed the 3-D basic state and spiral structure for atmospheric motion. The 3-D basic state results from the balance between the pressure gradient force, Coriolis force, and viscous force. The viscous force is very important in forming the spiral structure. In brief, the characteristics of the 3-D basic state are the cyclonic vorticity with updraft due to surface convergence and the anticyclonic vorticity with downdraft due to surface divergence. This is a typical atmospheric structure. Because of the nonlinear advective effect, the structure evolves continuously, and bifurcation of the structure can occur. In this paper, the stability of the vorticity and divergence field are discussed and the physical mechanism and conditions of bifurcation are analyzed.

2. The structure of divergence and vorticity field in the linear case

If the nonlinear advective terms are not considered, the dimensionless form of the horizontal dynamical equations and continuous equation (Holton, 1972; Houghton, 1985) can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial p}{\partial x} + \dfrac{1}{Ro}v - \dfrac{1}{Re}u \,, \\ \dfrac{\partial v}{\partial t} = -\dfrac{\partial p}{\partial y} - \dfrac{1}{Ro}u - \dfrac{1}{Re}v \,, \\ D + \dfrac{\partial w}{\partial z} = 0 \,, \end{cases} \tag{1}$$

where Ro and Re are the Rossby and Reynolds numbers, respectively. p is the non-dimensional perturbation pressure. In equation (1), the viscous force and thermal conduction use the frictional force form.

$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

denotes horizontal convergence ($D < 0$) or horizontal divergence ($D > 0$) (Charney, 1948; Andrews and Holton, 1987).

If we differentiate with respect to x and y respectively in the first and second equations of Eq.(1), then we can obtain the equations for the horizontal divergence D and vertical vorticity

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

(Lilly, 1982; Kurihara, 1982):

$$\begin{cases} \dfrac{\partial D}{\partial t} = -\dfrac{1}{Ro}\zeta_{\rm g} + \dfrac{1}{Ro}\zeta - \dfrac{1}{Re}D \,, \\ \dfrac{\partial \zeta}{\partial t} = -\dfrac{1}{Ro}D - \dfrac{1}{Re}\zeta \,, \end{cases} \tag{2}$$

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Fig. 1. The normal structures in the atmospheric motions: (a) low pressure; (b) high pressure.

where

$$\zeta_{\mathrm{g}} = Ro\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right)$$

is geostrophic vorticity, and $\zeta_{\mathrm{g}} > 0$ and $\zeta_{\mathrm{g}} < 0$ denote the centers of low and high pressure, respectively.

From Eq.(2), it is obvious that the viscous force plays the role to decrease the absolute value of horizontal divergence and vertical vorticity (Sullivan, 1959). The non-geostrophic vorticity $(\zeta - \zeta_{\mathrm{g}})$ is the major factor to cause the changes of divergence, when the viscous force is not considered. The horizontal convergence $(D < 0)$ increases the cyclonic vorticity, and decreases the absolute value of the anticyclonic vorticity. And all these result from the Coriolis force.

When the right hand of equation (2) is set to zero, the steady states of the divergence and vorticity field satisfy the following equations:

$$\begin{cases} -\dfrac{1}{Ro}\zeta_{\mathrm{g}} + \dfrac{1}{Ro}\zeta_{\mathrm{e}} - \dfrac{1}{Re}D_{\mathrm{e}} = 0\,, \\ -\dfrac{1}{Ro}D_{\mathrm{e}} - \dfrac{1}{Re}\zeta_{\mathrm{e}} = 0\,. \end{cases} \tag{3}$$

Equation (3) describes the structure resulting from the balance between the pressure gradient force, Coriolis force, and viscous force. From Eq.(3), one can see that if the surface air horizontally converges $(D_{\mathrm{e}} < 0)$, then $\partial w/\partial z > 0$ (obtained from the continuity equation). Because $w = 0$ at the surface, $\partial w/\partial z > 0$ means that the air is moving upwards $(w > 0)$. The cyclonic vorticity $(\zeta_{\mathrm{e}} > 0)$ is derived from the second equation of (3). So the vortex center is of low pressure $(\zeta_{\mathrm{g}} > 0)$ from the first equation of (3). Conversely, if the surface air horizontally diverges $(D_{\mathrm{e}} > 0)$, the air is moving downwards $(w < 0)$, so the anticyclonic vorticity $(\zeta_{\mathrm{e}} < 0)$ and high pressure of the vortex center are constructed(Panofsky and Dutton, 1984; Zeng, 1979). These two structures are often seen in the atmospheric motions (see Fig. 1) so they are the normal structures.

In fact, the steady divergence and vorticity under the balance of the three forces can be written as

$$\begin{cases} D_{\mathrm{e}} = -\dfrac{ReRo}{Ro^2 + Re^2}\zeta_{\mathrm{g}}\,, \\ \zeta_{\mathrm{e}} = \dfrac{Re^2}{Ro^2 + Re^2}\zeta_{\mathrm{g}}\,, \end{cases} \tag{4}$$

from which the following relation can be obtained

$$\zeta_{\mathrm{e}} = -\frac{Re}{Ro}D_{\mathrm{e}}\,. \tag{5}$$

From (4) and (5), we see that the sign of ζ_{g} is the same as that of ζ_{e}, but the signs of D_{e} and ζ_{g} are opposite. This result is consistent with the above analysis.

When we discard the viscous force $(Re \to \infty)$, then $D_{\mathrm{e}} \to 0$, $\zeta_{\mathrm{e}} \to \zeta_{\mathrm{g}}$, which is the basic behavior of 2-D geostrophic wind.

3. The structure and bifurcation of divergence and vorticity field in the nonlinear case

In Eq. (2), if we consider nonlinear terms resulting from the nonlinear advection for divergence and vorticity, then Eq. (2) can be written as

$$\begin{cases} \dfrac{\partial D}{\partial t} + D^2 = -\dfrac{1}{Ro}\zeta_{\mathrm{g}} + \dfrac{1}{Ro}\zeta - \dfrac{1}{Re}D\,, \\ \dfrac{\partial \zeta}{\partial t} + \zeta D = -\dfrac{1}{Ro}D - \dfrac{1}{Re}\zeta\,, \end{cases} \tag{6}$$

where D^2 and ζD are nonlinear terms, and next we will explain what role they play in the divergence and vorticity fields.

First, if we only consider the ζD term in the vorticity equation, then the variation of vorticity with time t is

$$\zeta = \zeta_0 e^{-\int_0^t D dt}\,, \tag{7}$$

where ζ_0 is the initial vorticity. The horizontal convergence $(D < 0)$ will increase the absolute value of cyclonic vorticity and anticyclonic vorticity. The horizontal divergence $(D > 0)$ decreases the absolute value of cyclonic vorticity and anticyclonic vorticity.

Second, if we only consider the D^2 term in the divergence equation, then the change of divergence with time t is

$$D = D_0 \frac{1}{1+D_0 t} , \quad (8)$$

where D_0 is the initial divergence. Hence, the nonlinear term decreases the absolute value of divergence.

In the following, we will discuss the bifurcation due to ζD and D^2, respectively.

If we only consider ζD, then Eq. (6) reduces to

$$\begin{cases} \dfrac{\partial D}{\partial t} = -\dfrac{1}{Ro}\zeta_{\rm g} + \dfrac{1}{Ro}\zeta - \dfrac{1}{Re}D , \\ \dfrac{\partial \zeta}{\partial t} = -\zeta D - \dfrac{1}{Ro}D - \dfrac{1}{Re}\zeta . \end{cases} \quad (9)$$

Setting the right hand of (9) to zero, the steady states (equilibrium states) satisfy the equations:

$$\begin{cases} -\dfrac{1}{Ro}\zeta_{\rm g} + \dfrac{1}{Ro}\zeta_{\rm s} - \dfrac{1}{Re}D_{\rm s} = 0 , \\ -\zeta_{\rm s} D_s - \dfrac{1}{Ro}D_{\rm s} - \dfrac{1}{Re}\zeta_{\rm s} = 0 , \end{cases} \quad (10)$$

which yield the following results:

$$\begin{cases} D_{\rm s} = \dfrac{Re}{Ro}(\zeta_{\rm s} - \zeta_{\rm g}) , \\ \zeta_{\rm s} = -\dfrac{ReD_{\rm s}}{Ro + RoReD_{\rm s}} , \end{cases} \quad (11)$$

from which it is easily seen that when $\zeta_{\rm g} > 0$, there are two equilibrium states:

$$P^{(1)}(D_{\rm s}^{(1)}<0,\ \zeta_{\rm s}^{(1)}>0),\ \ Q^{(1)}(D_{\rm s}^{(1)}<0, \zeta_{\rm s}^{(1)}<0) . \quad (12)$$

And when ζ_g<0, there are also two equilibrium states:

$$P^{(2)}(D_{\rm s}^{(2)}>0,\ \zeta_{\rm s}^{(2)}<0),\ \ Q^{(2)}(D_{\rm s}^{(2)}<0, \zeta_{\rm s}^{(2)}<0) . \quad (13)$$

It is obvious that $P^{(1)}$ and $P^{(2)}$ represent the normal cyclonic structure and anticyclonic structure, respectively (Chandrasekhar, 1961; Haltiner and Martin, 1957; Emmanuel, 1994), see Fig. 1 (a) and (b); and $Q^{(1)}$ and $Q^{(2)}$ represent anomalous structure due to the nonlinear term ζD.

In order to study the bifurcation of the equilibrium states, the Jacobi matrix of Eq. (9) in the equilibrium state, i.e.,

$$\boldsymbol{J}_1 \equiv \begin{pmatrix} -\dfrac{1}{Re} & \dfrac{1}{Ro} \\ -\dfrac{1}{Ro} - \zeta_{\rm s} & \dfrac{1}{Re} - D_{\rm s} \end{pmatrix} \quad (14)$$

is considered and its eigenvalue λ satisfies

$$\begin{vmatrix} -\dfrac{1}{Re} - \lambda & \dfrac{1}{Ro} \\ -\dfrac{1}{Ro} - \zeta_{\rm s} & \dfrac{1}{Re} - D_{\rm s} - \lambda \end{vmatrix} = 0 , \quad (15)$$

from which the following equation for λ can be obtained

$$\lambda^2 + \left(\frac{2}{Re} + D_{\rm s}\right)\lambda + \frac{1}{Ro^2} + \frac{1}{Re^2} + \frac{D_{\rm s}}{Re} + \frac{\zeta_{\rm s}}{Ro} = 0 . \quad (16)$$

It is obvious that whether λ is real or complex depends on Δ_1, i.e.,

$$\Delta_1 = \left(\frac{2}{Re} + D_{\rm s}\right)^2 - 4\left(\frac{1}{Ro^2} + \frac{1}{Re^2} + \frac{D_{\rm s}}{Re} + \frac{\zeta_{\rm s}}{Ro}\right) = D_{\rm s}^2 - \frac{4}{Ro}\left(\frac{1}{Ro} + \zeta_{\rm s}\right) . \quad (17)$$

For normal equilibrium state $P^{(1)}$, if $| D_s |< 1/Re$, then the constant term and the coefficient of the first order term in Eq. (16) are all positive, so the two roots of Eq. (16) are all either negative real or conjugate complex with a negative real part, so this means that the equilibrium state $P^{(1)}$ is stable. But when $| D_{\rm s} |> 1/Re$, then $\zeta_{\rm s} < 0$, and this is not the equilibrium state $P^{(1)}$.

For normal equilibrium state $P^{(2)}$, if $| \zeta_{\rm s} |< 1/Ro$, then the constant term and the coefficient of the first order term in Eq. (16) are all positive, so the two roots of Eq. (16) are all either negative real or conjugate complex with a negative real part, so this means that the equilibrium state $P^{(2)}$ is stable. But when $| \zeta_{\rm s} |> 1/Ro$, then $\Delta_1 > 0$ in Eq. (17), so the two roots of Eq. (16) are a positive one and a negative one, and the equilibrium state is unstable.

For anomalous equilibrium states $Q^{(1)}$ and $Q^{(2)}$, from Eq. (11), we can obtain

$$\zeta_{\rm s} = \frac{Re \mid D_{\rm s} \mid}{Ro(1 - Re \mid D_{\rm s} \mid)} .$$

Because of $| D_{\rm s} |> 1/Re$ (only this condition can result in $\zeta_{\rm s} < 0$), so $| \zeta_{\rm s} |> 1/Ro$, then $\Delta_1 > 0$ in Eq. (17), so the two roots of Eq. (16) are a positive one and a negative one, and the equilibrium state is unstable.

In summary, for cyclonic vorticity if $| D_{\rm s} |< 1/Re$ and for anticyclonic vorticity if $| \zeta_{\rm s} |< 1/Ro$, then the normal structures are all stable. And all anomalous structures are unstable. In the coordinates $(| D_{\rm s} |, Re)$ and $(| \zeta_{\rm s} |, Ro)$ with respect to equilibrium states and parameters, the bifurcation curves are shown in Fig. 2.

From Fig. 2, it is easily seen that the bifurcation curves are hyperbolic; below the curve $(| D_{\rm s} |= 1/Re)$ or $(| \zeta_{\rm s} |= 1/Ro)$, which denotes the critical equilibrium, the structures are normal, where updraft due to surface convergence and low pressure corresponds to cyclonic vorticity and downdraft due to divergence and high pressure corresponds to anticyclonic vorticity, respectively. Crossing this curve, there is convergence ($D_{\rm s} < 0$) with anticyclonic vorticity ($\zeta_{\rm s} < 0$); it is unstable and can easily evolve into a normal vortex structure through nonlinear advection (Kuznetsov, 1996; Kubicek and Marek, 1983).

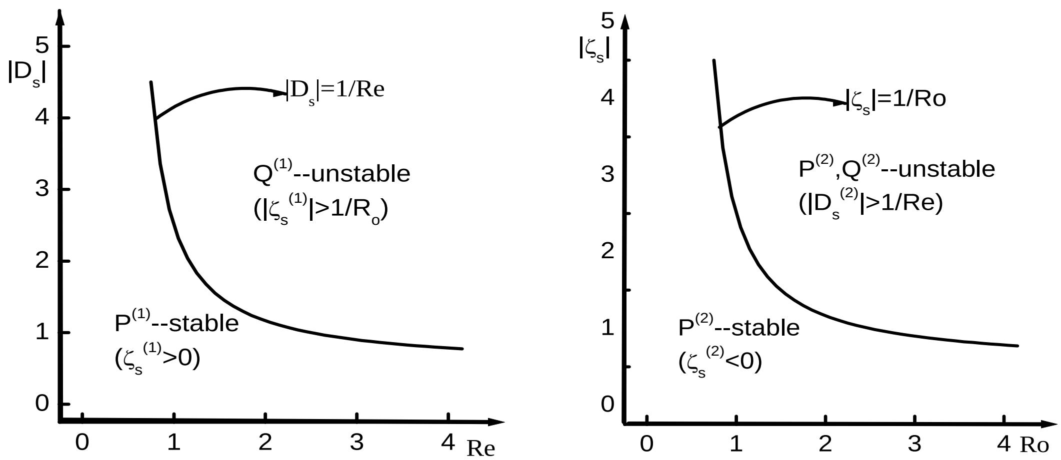


Fig. 2. The bifurcation curves under the action of ζD.

Next, the role of D^2 will be solely considered in Eq. (6), so Eq. (6) can be reduced to

$$\begin{cases} \dfrac{\partial D}{\partial t} = -D^2 - \dfrac{1}{Ro}\zeta_{\rm g} + \dfrac{1}{Ro}\zeta - \dfrac{1}{Re}D\,, \\ \dfrac{\partial \zeta}{\partial t} = -\dfrac{1}{Ro}D - \dfrac{1}{Re}\zeta\,. \end{cases} \tag{18}$$

Setting the right hand of both lines of Eq. (18) to zero yields the equilibrium states:

$$\begin{cases} -D_{\rm s}^2 - \dfrac{1}{Ro}\zeta_{\rm g} + \dfrac{1}{Ro}\zeta_{\rm s} - \dfrac{1}{Re}D_{\rm s} = 0\,, \\ -\dfrac{1}{Ro}D_{\rm s} - \dfrac{1}{Re}\zeta_{\rm s} = 0\,. \end{cases} \tag{19}$$

From the second equation of Eq. (19), one gets

$$\zeta_{\rm s} = -\frac{Re}{Ro}D_{\rm s}\,, \tag{20}$$

which indicates that the signs of $\zeta_{\rm s}$ and $D_{\rm s}$ are always opposite.

Eliminating the $\zeta_{\rm s}$ in Eq. (19) results in

$$D_{\rm s}^2 + \left(\frac{1}{Re} + \frac{Re}{Ro^2}\right)D_{\rm s} + \frac{\zeta_{\rm g}}{Ro} = 0\,; \tag{21}$$

when $\zeta_{\rm g} > 0$, there are two equilibrium states:

$$R^{(1)}(D_{\rm s}^{(1)} < 0, \zeta_{\rm s}^{(1)} > 0),\;\; R^{(2)}(D_{\rm s}^{(2)} < 0, \zeta_{\rm s}^{(2)} > 0)\,. \tag{22}$$

And when $\zeta_{\rm g} < 0$, there are also two equilibrium states:

$$S^{(1)}(D_{\rm s}^{(1)} > 0, \zeta_{\rm s}^{(1)} < 0),\;\; S^{(2)}(D_{\rm s}^{(2)} < 0, \zeta_{\rm s}^{(2)} > 0) \tag{23}$$

It is obvious that $R^{(1)}$ and $R^{(2)}$ represent normal cyclonic structure, $S^{(1)}$ represents normal anticyclonic strure.

In order to study the bifurcation of the equilibrium states, the Jacobi matrix of Eq. (18) in equilibrium state, i.e.

$$\boldsymbol{J}_2 \equiv \begin{pmatrix} -2D_s - \dfrac{1}{Re} & \dfrac{1}{Ro} \\ -\dfrac{1}{Ro} & -\dfrac{1}{Re} \end{pmatrix} \tag{24}$$

is considered and its eigenvalue λ satisfies

$$\begin{vmatrix} -2D_{\rm s} - \dfrac{1}{Re} - \lambda & \dfrac{1}{Ro} \\ -\dfrac{1}{Ro} & -\dfrac{1}{Re} - \lambda \end{vmatrix} = 0\,, \tag{25}$$

from which the following equation for λ can be obtained

$$\lambda^2 + \left(\frac{2}{Re} + 2D_{\rm s}\right)\lambda + \frac{1}{Ro^2} + \frac{1}{Re^2} + \frac{2D_{\rm s}}{Re} = 0\,. \tag{26}$$

It is obvious that whether λ is real or complex depends on Δ_2, i.e.,

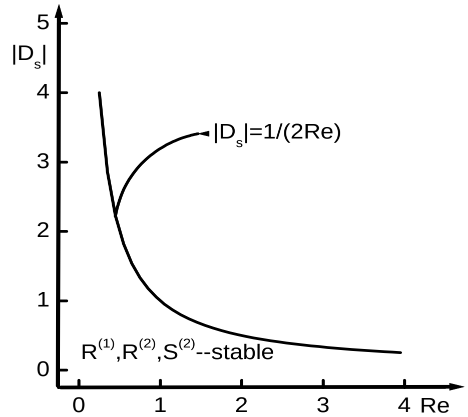
$$\begin{aligned} \Delta_2 &= \left(\frac{2}{Re} + 2D_{\rm s}\right)^2 - 4\left(\frac{1}{Ro^2} + \frac{1}{Re^2} + \frac{2D_{\rm s}}{Re}\right) \\ &= 4\left(D_{\rm s}^2 - \frac{1}{Ro^2}\right)\,. \end{aligned} \tag{27}$$

For equilibrium states $R^{(1)}$, $R^{(2)}$, and $S^{(2)}$, $D_{\rm s}$ is all negative, so there are three cases that need to be discussed:

(1) When $\mid D_{\rm s} \mid < 1/(2Re)$, from Eq. (26), it is easily determined that Eq. (26) has either negative real roots or conjugate complex roots with a negative real part, so the equilibrium states are stable.

(2) When $1/Re > \mid D_{\rm s} \mid > 1/(2Re)$, if $\mid D_{\rm s} \mid < 1/Ro$, from Eq. (26), the two roots of Eq. (26) are conjugate complex with a negative real part, so the equilibrium states are stable; but if $\mid D_{\rm s} \mid > 1/Ro$, from Eq. (26), the two roots of Eq. (26) are real with one positive and one negative, so the equilibrium states are unstable.

(3) When $\mid D_{\rm s} \mid > 1/Re$, the two roots of Eq. (26) have a positive real part, so the equilibrium states are unstable.



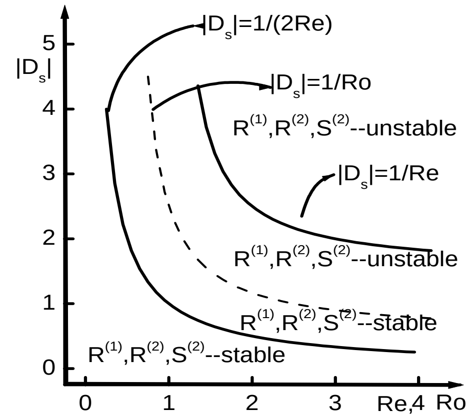


Fig. 3. The bifurcation curves under the action of D^2.

For equilibrium state $S^{(1)}$, the two roots of Eq. (26) have a negative real part, so the equilibrium states are stable.

So, the role of $D_{\rm s}$ is that the equilibrium states of downdraft due to the surface divergence and high pressure structure with anticyclonic vorticity are always stable; this is the normal structure. The stability for the updraft due to surface convergence and low pressure structure with cyclonic vorticity is shown in Fig. 3.

From Fig. 3, it is obvious that for the updraft due to convergence and low pressure equilibrium states $R^{(1)}$ and $R^{(2)}$ with cyclonic vorticity, if $|\ D_{\rm s}\ |< 1/(2Re)$ or $1/Re >|\ D_{\rm s}\ |> 1/(2Re)$ at the same time $|\ D_{\rm s}\ |< 1/Ro$, then the equilibrium states are stable. Here, we need to stress that the equilibrium $S^{(2)}$ is a high pressure center with convergent cyclonic vorticity, and it can also be stable.

4. Conclusion

The 3-D spiral structure resulting from the balance between the pressure gradient force, Coriolis force, and viscous force is a common atmospheric motion pattern. In this paper, nonlinear effects ζD and D^2 are considered. And it is shown that in the surface, the updraft due to convergence and low pressure with cyclonic vorticity and the downdraft due to divergence and high pressure with anticyclonic vorticity are stable under certain conditions. For this reason, they are the common atmospheric structures. At the same time, convergent structures with anticyclonic vorticity are always unstable, so generally they cannot be observed in the atmospheric motions. But, the convergent weak high pressure with cyclonic vorticity can exist in the atmosphere; it denotes the dying-out of the cyclones.

Acknowledgments. Many thanks are due to support from the Ministry of Science and Technology of China through a special public welfare project under Grant No. 2002DIB20070, and from the National Natural Science Foundation of China (Grant No. 40305006).

REFERENCES

Andrews, D. G., and J. R. Holton, 1987: *Middle Atmospheric Dynamics*, Academic Press, New York, 489pp.

Chandrasekhar, S., 1961: *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press, New York, 613pp.

Charney, J. G., 1948: On the scale of atmospheric motions. *Geophys. Publ.*, **17**, 1–17.

Emmanuel, K. A., 1994: *Atmospheric Convection*, Oxford University Press, New York, 580pp.

Haltiner, G. L., and F. L. Martin, 1957: *Dynamical and Physical Meteorology*. McGraw-Hill, New York, 470pp.

Holton, J. R., 1972: *An Introduction to Dynamic Meteorology*, Academic Press, New York, 319pp.

Houghton, H. G., 1985: *Physical Meteorology*, MIT, MA, Cambridge, 442pp.

Kubicek, M., and M. Marek, 1983: *Computational Methods in Bifurcation Theory and Dissipative Structure*, Springer-Verlag, Berlin, 276pp.

Kurihara, Y., 1982: Influence of environmental condition on the genesis of tropic storms. *Intense Atmospheric Vortices*, L. Bengtssoon and J. Lighthill, Eds., Springer-Verlag, Berlin, 71–79.

Kuznetsov, Y. A., 1996: *Elements of Applied Bifurcation Theory*. Springer-Verlag, Berlin, 551pp.

Lilly, D. K., 1982: The development and maintenance of rotation in convective storms. *Intense Atmospheric Vortices*, L. Bengtssoon and J. Lighthill, Eds., Springer-Verlag, Berlin, 149–160.

Liu Shida, Liu Shikuo, Fu Zuntao, Xin Guojun, and Liang Fuming, 2003: From 2-D geostrophic wind to 3-D vortex motions. *Chinese Journal of Geophysics*, **46**(4), 649–656.

Liu Shida, Xin Guojun, Liu Shikuo, and Liang Fuming, 2000: The 3-D spiral structure pattern in the atmosphere. *Adv. Atmos. Sci.*, **17**(4), 519–525.

Panofsky, H. A., and J. A. Dutton, 1984: *Atmospheric Turbulence*. John Wiley & Sons, New York, 397pp.

Sullivan, R. D., 1959: A two-cell vortex solution of the Navier-Stokes equations. *J. Aero/space Sci.*, **26**, 767–768.

Zeng Qingcun, 1979: *Mathematical and Physical Basis of Numerical Weather Forecasting* (*I*). Science Press, Beijing, 543pp.