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Uncertainty of the Numerical Solution of a Nonlinear System's Long-term Behavior and Global Convergence of the Numerical Pattern

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ABSTRACT

The computational uncertainty principle in nonlinear ordinary differential equations makes the numerical solution of the long-term behavior of nonlinear atmospheric equations have no meaning. The main reason is that, in the error analysis theory of present-day computational mathematics, the non-linear process between truncation error and rounding error is treated as a linear operation. In this paper, based on the operator equations of large-scale atmospheric movement, the above limitation is overcome by using the notion of cell mapping. Through studying the global asymptotic characteristics of the numerical pattern of the large-scale atmospheric equations, the definitions of the global convergence and an appropriate discrete algorithm of the numerical pattern are put forward. Three determinant theorems about the global convergence of the numerical pattern are presented, which provide the theoretical basis for constructing the globally convergent numerical pattern. Further, it is pointed out that only a globally convergent numerical pattern can improve the veracity of climatic prediction.

Key words: operator equation, uncertainty, appropriate discrete algorithm, global convergence

1. Introduction

The partial differential equations for atmospheric movement are often nonlinear and often very complex; we usually cannot obtain an analytic solution but must find a numerical solution. The general method is first to discretize the space variable and to turn the partial differential equations into ordinary differential equations, then we change the ordinary differential equations into an algebraic equation by discretizing the time variable. Finally, we can get the approximate numerical solution by solving a non-linear algebraic equation. As long as the numerical method is stable and convergent, the above process is completely feasible for short-term weather forecasts, however, for long-term forecasts and climate prediction, this process is far from enough. The main reason is the computational uncertainty principle in nonlinear ordinary differential equations (Li et al., 2000a, b). Li et al. (2000a, b) pointed out that because of the limitation of the computer precision when solving nonlinear ordinary differential equations by numerical methods, rounding error makes the numerical solution with any step-size unrelated to the real solution after the numerical integration within a certain time.

The main reason for the computational uncertainty principle is that we treat the nonlinear process between truncation error and rounding error as a linear operation. For a numerical method, we first prove the convergence with the assumption that the computation is exact (has no rounding error) and then consider restraining the impact of rounding error, i.e., to prove the stability of algorithm. For a linear system, the above method is completely feasible, but for a nonlinear system, though the independent effect of the two factors may be very small, their nonlinear interaction is not necessarily small. The total error may quickly accumulate with time, and ultimately it can easily make the numerical solution be independent of the real solution. So, for long-term forecasts and climatic prediction (when the timescale is more than one month or one season (Chou, 2002)), we must overcome

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this essential limitation and consider the nonlinear interaction between the truncation error and rounding error. We need to develop a global convergent numerical pattern (Chou, 2002).

In this paper, through studying the global asymptotic characteristics of the numerical pattern of the large-scale atmospheric equations by using the notion of cell mapping, we put forward the definitions of global convergence and an appropriate discrete algorithm of the numerical pattern. We also present three determinant theorems about the global convergence of the numerical pattern, which provide the theoretical basis for constructing the globally convergent numerical pattern. Further, we point out that only developing a globally convergent numerical pattern is the scientific method necessary to improve the veracity of long-term forecasts and that the theory of cell mapping is an effective method to study the nonlinear interaction between truncation error and rounding error.

2. Operator equation of large-scale atmospheric movement

If we use spherical coordinates, vector function

$$\varphi = (V_\lambda, V_\theta, \omega, \Phi, T)^{\mathrm{T}} ,$$

and operators $B, L,$ and N, the large-scale atmospheric equations can be written as the following operator equation:

$$B\frac{\partial\varphi}{\partial t} + [N(\varphi) + L]\varphi = \xi(t) , \quad (1)$$

$$B\varphi|_{t=t_0} = B\varphi_0 , \quad (2)$$

where

$$N(\varphi) = \begin{bmatrix} \Lambda & 2\Omega\cos\theta + \dfrac{\cot\theta}{a}V_\lambda & 0 & \dfrac{1}{a\sin\theta}\dfrac{\partial}{\partial\lambda} & 0 \\ -(2\Omega\cos\theta + \dfrac{\cot\theta}{a}V_\lambda) & \Lambda & 0 & \dfrac{1}{a}\dfrac{\partial}{\partial\theta} & 0 \\ 0 & 0 & 0 & \dfrac{\partial}{\partial p} & \dfrac{R}{p} \\ \dfrac{1}{a\sin\theta}\dfrac{\partial}{\partial\lambda} & \dfrac{1}{a\sin\theta}\dfrac{\partial}{\partial\theta}\sin\theta & \dfrac{\partial}{\partial p} & 0 & 0 \\ 0 & 0 & -\dfrac{R}{p} & 0 & \dfrac{R^2}{C^2}\Lambda \end{bmatrix}$$

$$B = \mathrm{diag}\left(1, 1, 0, 0, R^2/C^2\right) ,$$
$$L = \mathrm{diag}(L_1, L_1, 0, 0, L_2) ,$$
$$\xi(t) = (0, 0, 0, 0, R^2\varepsilon(t)/C^2C_p) ,$$

and

$$\Lambda = \frac{V_\lambda}{a\sin\theta}\frac{\partial}{\partial\lambda} + \frac{V_\theta}{a}\frac{\partial}{\partial\theta} + \omega\frac{\partial}{\partial p} ,$$
$$L_i = -\partial_p l_i \partial_p - \mu_i\nabla^2 ,$$
$$l_i = v_i(gp/R\overline{T})^2 \ \ (i = 1, 2) ,$$

$\xi(t)$ represents the non-constant external forcing case. For the other symbols, refer to Li and Chou (1997).

Because we discuss the global atmospheric movement, the resolution domain of (1)–(2) should be

$$\Omega = \{(\lambda, \theta, p)|0 \leqslant \lambda \leqslant 2\pi ,$$
$$0 \leqslant \theta \leqslant \pi, 0 \leqslant p \leqslant p_{\mathrm{s}}\} ,$$

where p_{s} is the ground atmospheric pressure, $0 < p_{\mathrm{s}} < \infty$. For the convenience of discussion, we do not consider the influence of the earth's surface and let p_{s}=1000 hPa, thus the boundary conditions of problems (1)–(2) can be written as:

On the earth's surface $p = p_{\mathrm{s}}$:

$$V_\lambda = V_\theta = \omega = 0 , \quad (3)$$

$$\frac{\partial T}{\partial p} = \alpha_{\mathrm{s}}(T_{\mathrm{s}} - T) ; \quad (4)$$

at the top of the atmosphere $p = 0$:

$$\frac{\partial V_\lambda}{\partial p} = \frac{\partial V_\theta}{\partial p} = 0 , \ \ \omega = 0 , \ \ \frac{\partial T}{\partial p} = 0 , \quad (5)$$

where $T_{\mathrm{s}} = T_{\mathrm{s}}(\lambda, \theta, t)$ is the temperature of the earth's surface, and α_{s} is a parameter which is related to the heat conductivity of turbulence and is dependent on the earth's surface characteristics.

Let H denote the collectivity of $\varphi = (V_\lambda, V_\theta, \omega, \Phi, T)^{\mathrm{T}}$ and assume that φ satisfies the periodic conditions and the boundary conditions (3)–(5), by Zeng (1979) we have $\varphi \in [L^2(\Omega)]^5$. If we define the inner product and norm of H as:

$$(\varphi_1, \varphi_2) = \int_{p_0}^{p_{\mathrm{s}}}\int_0^\pi\int_0^{2\pi} \varphi_1^{\mathrm{T}}\varphi_2 a^2 \sin\theta d\lambda d\theta dp , \quad (6)$$

$$\|\varphi\| = (\varphi, \varphi)^{\frac{1}{2}} , \tag{7}$$

then H is a Hilbert space. By Chou (1983) and Chou (1990), B and L are self-adjoint positive definite operators in Hilbert space H, i.e., $B = B^*$, $L = L^*, (B\varphi, \varphi) > 0$ and $(L\varphi, \varphi) > 0$ if $\varphi \neq 0$, where B^* and L^* are adjoint operators of B and L. N is a contra-adjoint operator in space H, i.e., $N = -N^*$, where N^* is an adjoint operator of N. Operator N generalizes the effects of the advection of rotating flow movement, the geostrophic departure power, the earth's spherical effect, and the barometric gradient power, etc. Operator L reflects the dissipative effect (Chou, 1990).

3. The long-term behavior obtained by the numerication integration is independent of the real solution

Because the form of equations (1)–(5) is very complex, we cannot obtain its analytic solution and we have no choice but to find its numerical solution. No matter what numerical method is used, the first thing to do is to discretize the space variable (λ, θ, p) and to change the partial differential equations (1)–(5) into ordinary differential equations. We still write these in the following operator form:

$$B_n \frac{\partial \varphi_n}{\partial t} + (N_n + L_n)\varphi_n = \xi_n(t) , \tag{8}$$

$$B_n \varphi_n|_{t=t_0} = B_n \varphi_n(0) , \tag{9}$$

where

$$\varphi_n = [f_1(t), f_2(t), ..., f_n(t)]^{\mathrm{T}} \in \tilde{R}_n$$

represents the change of atmospheric states with time, $\tilde{R}_n$ denotes the collectivity of n-dimensional functional vector φ_n. B_n, L_n and N_n are operators converted from the corresponding operators in (1), $\xi_n(t)$ is decided by the inhomogeneous term $\xi(t)$ and the boundary conditions. In order not to destroy the essential character of the original problem, the properties of the operators should be kept (Chou, 1983; Li and Chou, 1998), so B_n and L_n are positive definite symmetric matrixes of order n, N_n is an inverse symmetric matrix of order n.

The next problem is to find the numerical solution of the ordinary differential equations (8)–(9). The general method often used is to discretize the time variable and change (8)–(9) into an algebraic equation. Then by solving the nonlinear algebraic equation we can get the numerical solution of (8)–(9). As long as the numerical method used is stable and convergent, the above process is completely feasible for short-term weather forcasts, however, for long-term forcasts and climate prediction this process is far from enough. The reason is the following.

By theoretical analysis and numerical experiments for the initial value problem of the nonlinear ordinary differential equations of order one (Li et al., 2000a, b):

$$\frac{dy}{dt} = f(t, y) , \tag{10}$$

$$y|_{t=t_0} = y_0 , \tag{11}$$

Li et al. (2000a, b) present the computational uncertainty principle of nonlinear ordinary differential equations. They show that because of the limitation of the computer precision, when solving the problems (10) and (11) by using numerical methods, rounding error makes the numerical solution with any step-size unrelated to the real solution after a numerical integration of a certain time. And the numerical solution will not converge to the correct solution when the time step-size $h \to 0$. At the same time, due to the limited size of computer memory, there exist the maximum effective computational time and the optimal step-size (the step-size corresponding to the maximum effective computational time) for any numerical method. When the computational time exceeds the maximum effective computational time, the solution of the ordinary differential equation cannot be computed correctly by numerical methods. Improving the computer precision can slow down but cannot eliminate the impact of the rounding error.

The interaction of the truncation error and the rounding error of the numerical method is one of the most important reasons that induces the computational uncertainty principle. It is also the essential deficiency in the theory of the computational mathematics today. Two of the most important notions in computational mathematics are convergence and stability, which correspond to truncation error and rounding error respectively. The relation between convergence and stability should be very close, however, in the theory of error analysis of computational mathematics, we often discuss convergence and stability separately.

Equations (8)–(9) are actually a special form of (10)–(11), and their computation also has the above problems. So, when we want to give long-term numerical forecasts and numerical predictions of climates, we must overcome the essential deficiency in the theory of computational mathematics. The authors think that the proper scientific method is to study the properties of (1)–(5) for all initial values when $t \to \infty$. That is to say, we should study the global asymptotic characteristics of (1)–(5) and the global asymptotic characteristics of its numerical pattern. Then, by combining both, we can surely create more accurate predictions.

4. The global asymptotic characteristics of atmospheric patterns

We have the large-scale atmospheric operator equations (1)–(5) as examples, and we believe that it has a unique solution in Hilbert space H (Li and Chou, 1997).

4.1 *Global asymptotic characteristics of the partial differential equations*

By Li and Chou (2001), Wang et al. (1989), Li and Chou (1997b), for the operator equations (1)–(5) with bounded external forcing, there exists a bounded closed sphere B_K in H. The solution $\varphi(t)$ of the equations (1)–(5) with any initial value outside B_K will enter and stay forever in B_K when t is greater than some critical time τ. If we choose any initial value inside B_K, the solution $\varphi(t)$ will not go out of B_K.

Because the equations (1)–(5) has a unique solution (Li and Chou, 1997a; Chou, 1989), the solution $B\varphi(t)$ is uniquely decided by the initial value $B\varphi_0$, and the equations (1)–(5) actually define a mapping $S(t) : H \rightarrow H$ such that $S(t)B\varphi_0 = B\varphi(t)$. If we denote

$$S(t)R = \{S(t)B\varphi_0 | \forall B\varphi_0 \in R, R \subset H\} ,$$

then for any bounded set $R \subset H$, there exists $\tau(R) > 0$ such that $S(t)R \subset B_K$ for any $t \geqslant \tau(R)$, i.e., B_K is an attractive set. Let

$$A = \bigcap_{\alpha \geqslant 0} \overline{\bigcup_{t \geqslant \alpha} S\,(t)\,B_K} ,$$

then we know (Chou, 1990) that A is the functional invariant set of $S(t)$ (i.e., $S(t)A = A$ for any $t \geqslant 0$), and A is the global attractor of $S(t)$.

In other words, problem (1)–(5) has a global attractor A in the space H and A is an invariant point set. In spite of the initial states, the atmospheric system described by (1)–(5) will evolve into the state of the attractor set A with the increment of time. A reflects the final state of the system and is called the climate attractor. It reveals that the atmospheric system has a nonlinear adaptive process toward an external source and explains that the property of the dissipative structure is a fundamental characteristic of the atmospheric movement (Li and Chou, 1997c).

4.2 *Global asymptotic characteristics of the ordinary differential equations*

According to Lions et al. (1992) and Chou (1990), the Hausdorff dimension of climate attractor A is finite, and its estimated approximate value can be given. This means that, in space H, the limiting solution set of the atmospheric system will contract to a finite dimensional differential manifold. If the Hausdorff dimension of A can be obtained accurately, the atmospheric equations can be described precisely by a set of ordinary differential equations with finite dimension. However, nobody has obtained the ordinary differential equations yet. The usual process is to approximately describe (1)–(5) by using the ordinary differential equations (8)–(9). The result by this procedure is that the limiting solution set will not definitely include the finite dimensional manifold A no matter how big the space-resolving ratio is; the limiting solution set of (8) can only be viewed as the approximation of A.

In fact, by Li and Chou (2003) and Chou (1990), the operator equation (8) has a global attractor point set S_n with zero volume in space $\tilde{R}_n$. Set S_n depicts the global asymptotic character of the ordinary differential equations (8); it is the approximate form of the climate attractor A. The space-variable discrete algorithm of equation (1) determines the approximate degree.

4.3 *Global asymptotic characteristic of the numerical pattern*

Similarly, because the form of equations (8)–(9) is very complex, we cannot obtain the analytic form of point set S_n but must find its numerical form. That is to say, we must compute the numerical solution of (8)–(9) in space R_n. Suppose that we use the linear k-step difference method and substitute the differential quotient of time with the difference quotient of time in equations (8)–(9), then we get the difference equation:

$$D\{X(t_0 + m\tau), ..., X[t_0 + (m + k)\tau], \mu\} = 0 , \quad m = 0, 1, 2... \tag{12}$$

where $X[t_0 + (m+i)\tau]$ denotes the approximate value of state variable φ at time $t_0 + (m+i)\tau$ $(i = 0, ..., k)$, μ is a parameter which reflects the status of the external environment of the system, and $\tau = a\delta t$, where $a \geqslant 1$ is an integer and δt is the optimal step-size of the numerical computation (Li et al., 2000b). Equation (12) can be seen as the numerical pattern of the nonlinear systems (1)–(5). For convenience of discussion, let $k = 1$. From (12) we know that in R_n, the value of state variable X at time $t_0 + (m + 1)\tau$ is uniquely decided by its value at time $t_0 + m\tau$, i.e., there exists a mapping $F : R_n \rightarrow R_n$, such that

$$X[t_0 + (m + 1)\tau] = F[(X(t_0 + m\tau), \mu] , \quad m = 0, 1, 2, ... \tag{13}$$

In practical computation, because of the limited size of computer memory (we assume it is r), the rounding error causes the state variable X to no longer be a continuous variable; its true value is a discrete point sequence in space R_n. This course is called state variable X itself discretized. Thereby, the collectivity of X that the computer can actually store is not the whole space R_n but a finite point-set in R_n, which we denote as C_r. Then for any point $s \in C_r, s$ represents all the points in its neighborhood $U \in R_n$, where U is determined by s and r. In other words, because of the limitation of computer precision, all the points in neighborhood U have been rounded as point s, so s is actually a cell in R_n and there are an infinite number of points in the cell. Namely, a point in C_r corresponds to a cell in R_n, and the collection of all cells is not the whole space R_n but only a subset of R_n.

Hence, the mapping $F : C_r \to C_r$ defined by (13) is actually a cell mapping (Hsu, 1987). The different values of the numerical solution at different times is just the transformation from one cell to another cell. If a cell returns to itself after several mappings, the cell is called a periodic cell, otherwise, it is called a temporary state cell. The temporary state cell only has a temporary meaning. Assume $P_r(n)$ denotes the entire set of periodic cells of mapping $F : C_r \to C_r$, then by (13) and the conception and theory of cell mapping (Hus, 1987), we know that $P_r(n)$ represents the global asymptotic character of the numerical solution of equation (8).

Namely, for all initial values, the numerical solution of operator equation (8) will evolve into the attractive point-set $P_r(n)$ when $t \to \infty$. $P_r(n)$ is just the approximate form of the global attractive point-set S_n; the approximate degree is determined by the time-variable discrete algorithm of equation (8) and the whole computing process. The structure and statistical characteristics of $P_r\,(n)$ are the climate characteristics of the numerical pattern (12).

In summarily, we first get the climate attractor A by the global asymptotic characteristies of the atmospheric movement equation, then we get the approximate form S_n of the climate attractor A by the discretization of the space variable, and finally we get the approximate form $P_r(n)$ of S_n by discretization of the time variable. As we solve the problems, the spaces related to the problems are gradually simplified, from the initial uncountable space H with infinite dimension to uncountable space $\tilde{R}_n$ with finite dimension, then to countable space R_n with finite dimension, and finally, to the set C_r with a finite number of points in space R_n. The simplification brings new convenience to the research, but it also introduces a new problem–the degree of approximation. In the following secction, we discuss in detail how to measure the approximate degree of one numerical model for the original system.

5. New concepts—global convergence and appropriate discretization

For nonlinear systems (1)–(5), by the above process, we get three point sets: A, S_n, and $P_r(n)$. They are unrelated to the initial values and are decided by the external parameters and algorithm. If A can depict the practical climate more exactly, we hope that $P_r(n)$ can also approximate A very well.

For measuring the approximate degree of $P_r(n)$ and A, we introduce the Hausdorff distance between two sets in Hilbert space. Supposing B and C are two arbitrary sets in H, the Hausdorff distance between them is defined as:

$$D_{\mathrm{H}}(B, C) = \max\{D(B, C), D(C, B)\}\ , \qquad (14)$$

where

$$D(B, C) = \sup_{b\in B} \inf_{c\in C} \rho(b, c)\ ,$$

where $\rho(b, c)$ is the distance between the variables b and c. It is obvious that $D_{\mathrm{H}}(B, C) = 0$ if and only if $B = C$.

Definition 1 Let A denote the finite-dimensional attractive set in H of the partial differential equations (1)–(5). S_n denotes the attractive set with zero volume in $\tilde{R}_n$ of the ordinary differential equations (8) which are derived from (1)–(5) when the space variable is discretized according to some discrete method. If $\lim\limits_{n\to\infty} D_{\mathrm{H}}(S_n, A) = 0$, we call the solution of (8) globally convergent. The discrete method that makes the solution of (8) globally convergent is called the $H \to \tilde{R}_n$ appropriate discrete algorithm.

Definition 2 Supposing S_n is the attractive set with zero volume in $\tilde{R}_n$ of the ordinary differential equations (8), $P_r(n)$ is the attractive set of the difference equation (13) when it is solved with optimal step-size δt in a computer with r bytes of memory. If $\lim\limits_{r\to\infty} D_{\mathrm{H}}[P_r(n), S_n] = 0$, we call the solution of (13) globally convergent. The discrete method that makes the solution of (13) globally convergent is called the $\tilde{R}_n \to R_n$ appropriate discrete algorithm.

Definition 3 Let A denote the finite-dimensional attractive set in H of the partial differential equations (1)–(5). After the discretization of the variables of space, time, and state, (1)–(5) become a climate numerical pattern whose attractive set is $P_r(n)$. If $\lim\limits_{r\to\infty, n\to\infty} D_{\mathrm{H}}[P_r(n), A] = 0$, we call the climate numerical pattern globally convergent. The discrete method which makes the climate numerical pattern globally convergent is called the $H \to R_n$ appropriate discrete algorithm.

From the above definitions, we know that only if $H \to \tilde{R}_n$ is an appropriate discrete algorithm, then the set S_n obtained by ordinary differential equations (8) will not distort the essential character of the climatic attractor A. And only if $\tilde{R}_n \to R_n$ is an appropriate discrete algorithm, then the attractive set $P_r(n)$ obtained by the difference equation (13) will approximate set S_n well in the time series. In other words, only the globally convergent numerical model can approximately describe the change of practical climate. Next, we will present the determinant theorems for a globally convergent numerical pattern.

Theorem 1 When we construct a climatic numerical model from the partial differential equations, if $H \to \tilde{R}_n$ is not an appropriate discrete algorithm and $\tilde{R}_n \to R_n$ is an appropriate discrete algorithm, then $H \to R_n$ also is not an appropriate discrete algorithm, i.e., the climate numerical pattern is not globally convergent.

Proof: If $H \to \tilde{R}_n$ is not an appropriate discrete algorithm, then from Definition 1, we know that given $\varepsilon_0 > 0$, then for any N there exists $n_0 > N$ such that $D_{\mathrm{H}}(S_{n_0}, A) \geqslant \varepsilon_0$, i.e.,

$$\lim_{n\to\infty} D_{\mathrm{H}}(S_n, A) \neq 0 \, , \tag{15}$$

and if $\tilde{R}_n \to R_n$ is an appropriate discrete algorithm, then from Definition 2, we have

$$\lim_{r\to\infty} D_{\mathrm{H}}[P_r(n), S_n] = 0 \, , \tag{16}$$

It is easy to prove that the Hausdorff distance defined by (14) satisfies the distance axiom, then we have

$$D_{\mathrm{H}}[P_r(n), A] \geqslant |D_{\mathrm{H}}(S_n, A) - D_{\mathrm{H}}[P_r(n), S_n]| \, .$$

To obtain the limit about r for both sides of the above inequality, then from (16) we have

$$\lim_{r\to\infty} D_{\mathrm{H}}[P_r(n), A] \geqslant D_{\mathrm{H}}(S_n, A) \, .$$

Also, from (15) we know

$$\lim_{r\to\infty, n\to\infty} D_{\mathrm{H}}[P_r(n), A] \neq 0 \, , \tag{17}$$

then according to Definition 3, $H \to R_n$ is not an appropriate discrete algorithm and the climate numerical pattern is not globally convergent.

Theorem 2 When we construct a climatic numerical model from the partial differential equations, if $H \to \tilde{R}_n$ is an appropriate discrete algorithm and $\tilde{R}_n \to R_n$ is not an appropriate discrete algorithm, then $H \to R_n$ also is not an appropriate discrete algorithm, i.e., the climate numerical pattern is not globally convergent.

Proof: The course of the proof is the same as for Theorem 1.

Theorem 3 When we construct a climatic numerical model from the partial differential equations, if $H \to \tilde{R}_n$ and $\tilde{R}_n \to R_n$ are both appropriate discrete algorithms, then $H \to R_n$ is also an appropriate discrete algorithm, i.e., the climate numerical pattern is globally convergent.

Proof: If $H \to \tilde{R}_n$ is an appropriate discrete algorithm, then from Definition 1, we have

$$\lim_{n\to\infty} D_{\mathrm{H}}(S_n, A) = 0 \, .$$

In the same way, if $\tilde{R}_n \to R_n$ is an appropriate discrete algorithm, then from Definition 2, we have

$$\lim_{r\to\infty} D_{\mathrm{H}}[P_r(n), S_n] = 0 \, .$$

From the distance axiom, the following triangle inequality is true

$$D_{\mathrm{H}}[P_r(n), A] \leqslant D_{\mathrm{H}}(S_n, A) + D_{\mathrm{H}}[P_r(n), S_n] \, .$$

To obtain the limit of both sides of the above triangle inequality, and by the non-negativity of distance, we have

$$0 \leqslant \lim_{r\to\infty, n\to\infty} D_{\mathrm{H}}[P_r(n), A] \leqslant \lim_{n\to\infty} D_{\mathrm{H}}(S_n, A) + \lim_{n\to\infty}\{\lim_{r\to\infty} D_{\mathrm{H}}[P_r(n)\, , S_n]\} = 0 \, . \tag{18}$$

Then

$$\lim_{r\to\infty, n\to\infty} D_{\mathrm{H}}[P_r(n), A] = 0 \, .$$

So, according to Definition 3, $H \to R_n$ is an appropriate discrete algorithm and the climate numerical pattern is globally convergent.

In conclusion, in the process of constructing the climatic numerical pattern, if the discrete-space approximation is not appropriate discrete or if the discrete-time approximation is not appropriate discrete, then the climatic numerical pattern is not appropriate discrete and the numerical solution is also not globally convergent. Only if $H \to \tilde{R}_n$ and $\tilde{R}_n \to R_n$ are both appropriate discrete algorithms, the climatic numerical pattern is globally convergent.

6. Conclusions and new problems

Although meteorologists have done a lot of very important work, we still have no accepted, good numerical pattern for short-term or mid-term climate prediction (Chou, 2002). Maybe one of the reasons is that we do not attach enough importance to the essential characters of different atmospheric phenomena and, for a given numerical pattern, we hardly pre-analyze the reliability of the numerical result in theory. If it is possible, we should try to develop different mathematical models and construct different globally convergent numerical patterns for different predictions. For each prediction we need to try to do the following two things. The first is to analyze the global asymptotic characteristics of both the mathematical models and numerical pattern. According to the ideas in this paper, using the existing theories and methods of mathematics, we should first try to analyze the global asymptotic characteristics of the mathematical models and then begin to construct the globally convergent numerical pattern. In present-day mathematical theories, there are many good results and methods regarding the global behavior of partial differential equations. Applying them may help us to select the numerical methods with clearer aims, and it also may help us to numerically reflect the essential characters of the original problems. The second is, for a given numerical pattern, we need to estimate the maximum effective computational time and analyze the reliability of the numerical results. Because different numerical patterns with a given computational precision have different effective time scales (Li et al., 2000a), we should try to insure that the numerical results can really approximate the real problem under investigation. Before prediction, we need to consider the influence of the rounding error and truncation error together, i.e., we need evaluate the error $D_{\mathrm{H}}[P_r(n), A]$ in theory. If $D_{\mathrm{H}}[P_r(n), A]$ increases after some computing time, the numerical result is no longer reliable, and we need some correction to ensure that $D_{\mathrm{H}}[P_r(n), A]$ is small, i.e., we need to always try to keep the global convergence of the numerical pattern during the whole computing procedure.

In conclusion, developing a globally convergent numerical pattern is the appropriate scientific method to improve the veracity of prediction. In this paper, by using mathematical methods and cell mapping theory, we study the global asymptotic properties of the numerical pattern of the large-scale atmospheric equations. Then, we put forward the definitions of global convergence and appropriate discrete algorithm of the numerical pattern and three determinant theorems regarding the global convergence of the numerical pattern. They provide the theoretical basis for constructing a globally convergent numerical pattern.

However, the authors think that only seeking the appropriate discrete method for both the space and time variables is not the whole problem. By Li et al. (2000b), because of the limitation of computer precision, there exists an optimal step-size h to numerically solve the initial value problem of a nonlinear ordinary differential equation. Among h, the order of the numerical method, and the amount of computer memory r, there exists a common relation which is independent on the type of the equation, initial value, and method itself. That in fact means, since the whole error is the co-impact of the rounding error and truncation error, it is optimal when the two errors are in correspondence. And, with definite computer precision, minimizing the time step-size beyond the limitation will not decrease the whole error. In the same way, for partial differential equations (1)–(5), the error $D_{\mathrm{H}}[P_r(n), A]$ between the solution $P_r(n)$ of the numerical model and the real solution A is the superimposition of the errors of the discretization of both space and time. It is optimal when the two errors are in correspondence. With some definite computer precision, there should exist an optimal n. And there also exists a common relationship among the computer precision r (i.e., r denotes the maximum length of rational real numbers that the computer can rearly store.) time step-size h, and space resolution ratio n; and only if the relationship is satisfied, the climatic numerical model derived by appropriate discretizations of both the space and the time variables is appropriate discrete. Otherwise, when $r \to \infty, n \to \infty$, the solution of the numerical model may not be globally convergent. The work on this aspect needs further development and we think that it will be a new domain worth exploring.

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