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Generalized Method of Variational Analysis for 3-D Flow

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ABSTRACT

The generalized method of variational analysis (GMVA) suggested for 2-D wind observations by Huang et al. is extended to 3-D cases. Just as in 2-D cases, the regularization idea is applied. But due to the complexity of the 3-D cases, the vertical vorticity is taken as a stable functional. The results indicate that wind observations can be both variationally optimized and filtered. The efficiency of GMVA is also checked in a numerical test. Finally, 3-D wind observations with random disturbances are manipulated by GMVA after being filtered.

Key words: regularization methods , inverse problem, generalized method of variational analysis, filtering

1. Introduction

A forecasting system is concerned with data objective analysis which is usually classified into two types. The first kind is to interpolate the data onto grid points, i.e. establish the statistical structure of the physical fields by using abundant observational data, obtain the weight of every physical field, and interpolate the observational data onto grid points. This provides the initial fields (Zeng, 1997). But initial fields often contain high frequency noise, which will been enlarged in a numerical integration. Hence, the initial field needs to be adjusted to filter the noise, which is the so-called initialization. The second kind of data objective analysis is variational analysis (VA), which was first presented for 2-D flow by Sasaki in the 1950s, and in essence it is an application of conditional variation to numerical weather forecasting, with a concentration on the analysis of the wind field, flow field, and so on (Sasaki, 1969, 1970). It is on the basis of the above work that Le Dimet and Talagrand (1986) and Talagrand and Curtier (1987) proposed the idea of data variational assimilation in the 1980s, which markedly enhanced the forecasting precision by obtaining the optimal initial field.

It is known that VA is a powerful tool in most cases. However, Huang et al. pointed out that, for 2-D wind observations with high frequency oscillations, though an optimal analyzed field can be extracted from the wind observations through VA, the discrepancies between the observations and the analyzed field is “not small” and the latter still contains high frequency components. In view of this fact, they developed a generalized method of variational analysis (GMVA) incorporating the regularization technique. An ideal numerical test shows that 2-D wind observations containing high frequency noise can be both variationally optimized and filtered.

In this paper, the GMVA will be extended to the case of 3-D wind observations. First, the method of variational analysis (MVA) for 2-D wind observations by Sasaki is applied to ideal 3-D wind observations with high frequency oscillations, and its deficiencies will also be pointed out. And then, the GMVA for 3-D wind observations with the aid of the regularization idea is introduced. Because of the intrinsic difference between the 2-D and 3-D wind field, the GMVA for 3-D wind observations is not a simple extension of MVA for 2-D wind observations.

This paper is organized as follows. Section 2 is a review and comment of Sasaki’s MVA. In section 3, the GMVA with the regularization technique is introduced. Section 4 gives an example of numerical results through the GMVA and the comparison with Sasaki’s results. The paper ends with a concluding remark in section 5.

2. Review of Sasaki’s method of variational analysis

The flow field is assumed to be three dimensional with the domain $\Omega \subset R^3$ and the boundary $\partial\Omega$. Because of observational error, real wind does not satisfy the incompressible theorem, i.e.,

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} \neq 0 \,. \tag{2.1}$$

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So an analysed wind field $\boldsymbol{V}=(u,v,w)$ is designed to satisfy the incompressible theorem, i.e.,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\,, \tag{2.1*}$$

and to make the following cost functional minimal.

$$J[u,v,w]=\int_\Omega[(u-\tilde{u})^2+(v-\tilde{v})^2+(w-\tilde{w})^2]dxdydz=\min\ . \tag{2.2}$$

This is a conditional variation problem. Introducing a Lagrange multiplier $\lambda(x,y,z)$, the new cost functional is

$$J[u,v,w,\lambda]=\int_\Omega\left[(u-\tilde{u})^2+(v-\tilde{v})^2+(w-\tilde{w})^2-2\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]dxdydz=\min \tag{2.2*}$$

Setting $\delta J=0$ leads to three equations

$$u=\tilde{u}-\frac{\partial\lambda}{\partial x}\,,\ \ v=\tilde{v}-\frac{\partial\lambda}{\partial y}\,,\ \ w=\tilde{w}-\frac{\partial\lambda}{\partial z}\,. \tag{2.3}$$

Here the boundary condition is taken as $\lambda|_{\partial\Omega}=0$. It follows from (2.1*) and (2.3) that

$$\Delta\lambda=\frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{v}}{\partial y}+\frac{\partial\tilde{w}}{\partial z}\,,\ \ \lambda|_{\partial\Omega}=0\,. \tag{2.4}$$

Eq. (2.4) is the Dirichlet problem of the Poission equation, which can be solved by the Successive Over-Relaxation (SOR) method. The optimal analysed value (u,v,w) can then be obtained from Eq. (2.3).

Now an example is specified to show the deficiency of Sasaki's MVA for 3-D wind observations. Suppose $(u_\mathrm{t},v_\mathrm{t},w_\mathrm{t})$ is a wind true field which has the form

$$u_\mathrm{t}=\cos(x)sh(y)\sin(z),\ \ v_\mathrm{t}=\sin(x)ch(y)\sin(z),$$
$$w_\mathrm{t}=0.5\,. \tag{2.5}$$

It satisfies (2.1*) by all appearances. The domain is $\Omega=(0,\pi)\times(0,\pi)\times(0,\pi)$, and an observational data point $(\tilde{u},\tilde{v},\tilde{w})$ is assumed to be

$$\begin{aligned}\tilde{u}&=u_\mathrm{t}+\varepsilon\cos(nx)\sin(my)\sin(z)\,,\\ \tilde{v}&=v_\mathrm{t}+\varepsilon\sin(nx)\cos(my)\sin(z)\,,\\ \tilde{w}&=w_\mathrm{t}\,,\end{aligned} \tag{2.6}$$

where ε is very small positive number, and n,m are positive integers, the second term in $\tilde{u}$ and $\tilde{v}$ are small magnitude disturbances superimposed on the true flow field $(u_\mathrm{t},v_\mathrm{t},w_\mathrm{t})$. The larger $n(m)$ is, the smaller the space scale of disturbances in the $x(y)$ direction is. So

$$\left\{\begin{aligned}\|\tilde{u}-u_\mathrm{t}\|_{H_1}&=\left\{\int_\Omega\left[(\tilde{u}-u_\mathrm{t})^2+|\nabla\tilde{u}-\nabla(u_\mathrm{t})|^2\right]dxdydz\right\}^{1/2}=\varepsilon\sqrt{m^2+n^2+2}\left(\frac{\pi}{2}\right)^{3/2}\,,\\ \|\tilde{v}-v_\mathrm{t}\|_{H_1}&=\left\{\int_\Omega\left[(\tilde{v}-v_\mathrm{t})^2+|\nabla\tilde{v}-\nabla(v_\mathrm{t})|^2\right]dxdydz\right\}^{1/2}=\varepsilon\sqrt{m^2+n^2+2}\left(\frac{\pi}{2}\right)^{3/2}\,,\\ \|\tilde{w}-w_\mathrm{t}\|_{H_1}&=\left\{\int_\Omega\left[(\tilde{w}-w_\mathrm{t})^2+|\nabla\tilde{w}-\nabla(w_\mathrm{t})|^2\right]dxdydz\right\}^{1/2}=0\,,\end{aligned}\right. \tag{2.7}$$

where $\|\bullet\|_{H_1}$ is a norm in Soblev space H_1

$$\|\phi\|_{H_1}=\left\{\int_\Omega\left[|\phi|^2+|\nabla\phi|\right]^2dxdydz\right\}^{1/2}\,,$$

and this can reflect the deviation of the observations from the true wind field. As shown in (2.7), $\|\tilde{u}-u_\mathrm{t}\|_{H_1}$, $\|\tilde{v}-v_\mathrm{t}\|_{H_1}$, and $\|\tilde{u}-u_\mathrm{t}\|_{H_1}$ will also be small enough as long as ε is small.

For the observational data, Eqs. (2.6) and (2.4) become

$$\Delta\lambda=\frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{v}}{\partial y}+\frac{\partial\tilde{w}}{\partial z}=-\varepsilon(n+m)\sin(nx)\sin(my)\sin(z)\,,\ \ \lambda|_{\partial\Omega}=0 \tag{2.8}$$

The Dirichlet problem of Poission equation (2.8) has a unique solution, which can be sought by the method of undetermined coefficients. Let

$$\lambda=A\sin(nx)\sin(my)\sin(z)\,.$$

Here, A is a constant to be determined. Then

$$\lambda=\frac{\varepsilon(n+m)}{n^2+m^2+1}\sin(nx)\sin(my)\sin z\,. \tag{2.9}$$

The difference between the analysis value (u,v,w) and the true wind field is $(u_\mathrm{t},v_\mathrm{t},w_\mathrm{t})$ is

$$\left\{\begin{aligned}\|\tilde{u}-u_\mathrm{t}\|_{H_1}&=\varepsilon\sqrt{m^2+n^2+2}\left(\frac{\pi}{2}\right)^{3/2}\frac{m^2-nm+1}{n^2+m^2+1}=\varepsilon O(n)\,,\ m=O(n)\,,\ \text{but}\ \ m\neq n,n,m\to\infty\,,\\ \|\tilde{u}-u_\mathrm{t}\|_{H_1}&=\varepsilon\sqrt{m^2+n^2+2}\left(\frac{\pi}{2}\right)^{3/2}\frac{m^2-nm+1}{n^2+m^2+1}=\varepsilon O(n)\,,\ m=O(n)\,,\ \text{but}\ \ m\neq n,n,m\to\infty\,,\\ \|\tilde{w}-w_\mathrm{t}\|_{H_1}&=\varepsilon\sqrt{m^2+n^2+2}\left(\frac{\pi}{2}\right)^{3/2}\frac{n+m}{n^2+m^2+1}=\varepsilon O(l)\,,\ m=O(n)\,,\ \text{but}\ \ m\neq n,n,m\to\infty\,,\end{aligned}\right. \tag{2.10}$$

It is easily seen from Eq. (2.6) that the observational wind possesses high frequency components for sufficiently large n and m. From Eq. (2.10), using Sasaki's MVA, the vertical component w of an analyzed wind field is of minor discrepancy, but the horizontal components (u, v) are of large discrepancy and therefore are not improved. Thus, Sasaki's MVA needs to be studied further.

3. Generalized method of variational analysis

According to the regularization idea (Tikhonov and Arsenin, 1977; Kirsch, 1996), a stable functional is introduced in the cost functional $J[u, v, w]$ so as to restrain high frequency noise (just as filtering) and to ensure the uniqueness of the solution of the variational problem (Huang et al., 2002, 2003). Here the stable cost functional is specified as

$$\int_\Omega \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)^2 dxdydz$$

where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

is the vertical vorticity. The reason for choosing the vertical vorticity ζ as the stable functional is as follows: the regularization stable functional must contain as much important physical information as possible. In dynamical meteorology, atmospheric motion is quasi-horizontal, i.e., the vertical component of the vorticity is leading. So the vertical vorticity is taken as the stable functional. Otherwise, taking the whole vorticity (including the x, y, z components of the vorticity) as the stable functional will cause the following aftermath. The problem becomes so complicated that the boundary conditions are difficult to determine, and the final analysis field is not satisfactory.

The new cost function is then

$$\begin{aligned}\tilde{J}[u, v, w] = \int_\Omega &\left[(u-\tilde{u})^2 + (v-\tilde{v})^2 + (w-\tilde{w})^2\right.\\ &- 2\lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dxdydz\\ &+ \gamma\int_\Omega \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)^2 dxdydz = \min\,,\end{aligned} \tag{3.1}$$

where $\gamma > 0$ is a regularization parameter. Due to $\frac{1}{2}\delta\tilde{J} = 0$, the following form can be given

$$\begin{aligned}0 &= \int_\Omega \Big[\, (u-\tilde{u})\delta u + (v-\tilde{v})\delta v + (w-\tilde{w})\delta w\\ &\quad + \lambda\left(\frac{\partial \delta u}{\partial x} + \frac{\partial \delta v}{\partial y} + \frac{\partial \delta w}{\partial z}\right) dxdydz\\ &\quad + \gamma\int_\Omega \zeta\left(\frac{\partial \delta v}{\partial x} - \frac{\partial \delta u}{\partial y}\right)\Big] dxdydz\\ &= \int_\Omega \left[\left(u - \tilde{u} + \frac{\partial\lambda}{\partial x} + \gamma\frac{\partial\zeta}{\partial y}\right)\delta u\right.\\ &\quad + \left(v - \tilde{v} + \frac{\partial\lambda}{\partial y} - \gamma\frac{\partial\zeta}{\partial x}\right)\delta v\\ &\quad + \left.\left(w - \tilde{w} + \frac{\partial\lambda}{\partial z}\right)\delta v\right] dxdydz\\ &\quad - \int_\Omega \left[\frac{\partial(\lambda\delta u)}{\partial x} + \frac{\partial(\lambda\delta v)}{\partial y} + \frac{\partial(\lambda\delta w)}{\partial z}\right] dxdydz\\ &\quad + \gamma\int_\Omega \left[\frac{\partial(\zeta\delta v)}{\partial x} - \frac{\partial(\zeta\delta u)}{\partial y}\right] dxdydz\,.\end{aligned} \tag{3.2}$$

Using the Green's formula, (2.2) changes to

$$\begin{aligned}0 = \int_\Omega &\left[\left(u - \tilde{u} + \frac{\partial\lambda}{\partial x} + \gamma\frac{\partial\zeta}{\partial y}\right)\delta u\right.\\ &+ \left(v - \tilde{v} + \frac{\partial\lambda}{\partial y} - \gamma\frac{\partial\zeta}{\partial x}\right)\delta v\\ &\left.\left(w - \tilde{w} + \frac{\partial\lambda}{\partial z}\right)\delta v\right] dxdydz\\ &- \int_\Omega \lambda(\delta u, \delta v, \delta w)\cdot \boldsymbol{n} ds\\ &+ \gamma\int_\Omega (\zeta\delta v, -\zeta\delta u, 0)\cdot\boldsymbol{n} ds = 0\,.\end{aligned} \tag{3.3}$$

Here $\boldsymbol{n}$ is the outward unit normal on the boundary $\partial\Omega$. The arbitrariness of $\delta u, \delta v$, and δw gives rise to the following Euler equation

$$\begin{cases} u = \tilde{u} - \dfrac{\partial\lambda}{\partial x} - \gamma\dfrac{\partial\xi}{\partial y}\,,\\ v = \tilde{v} - \dfrac{\partial\lambda}{\partial y} + \gamma\dfrac{\partial\xi}{\partial x}\,,\\ w = \tilde{w} - \dfrac{\partial\lambda}{\partial z}\,. \end{cases} \tag{3.4}$$

Equation (3.4) satisfy the condition on $\partial\Omega$

$$-\int_\Omega \lambda(\delta u, \delta v, \delta w)\cdot\boldsymbol{n} ds + \gamma\int_\Omega \zeta(\delta v, -\delta u, 0)\cdot\boldsymbol{n} ds = 0\,. \tag{3.5}$$

We consider

$$\lambda|_{\partial\Omega} = 0 \quad \text{and} \quad \zeta(\delta v, -\delta u, 0)|_{\partial\Omega} = 0\,.$$

Now that (u, v, w) in Eq. (3.4) satisfy Eq. (2.1*), then λ satisfies Poission Eq. (2.4) and can be calculated.

On the other hand,

$$\Delta\zeta-\frac{\zeta}{\gamma}=-\frac{\tilde{\zeta}}{\gamma}\,,\quad \int_{\Omega}\zeta(\delta v,-\delta u,0)\cdot\boldsymbol{n}ds=0\,,\quad (3.6)$$

where $\tilde{\zeta}$ is the vertical vorticity of the observational wind.

Now the boundary condition can be classified into two cases as $\delta(v,-u)\cdot\boldsymbol{n}|_{\partial\Omega}=0$ and $\zeta|_{\partial\Omega}=0$. These will be respectively discussed below.

Case 1 The boundary condition has the form of $\zeta|_{\partial\Omega}=0$, so $\zeta(x,y,z)$ can be determined from Eq. (3.6).Then the analysis value (u,v,w) can be obtained after substituting (λ,ζ) into Eq .(3.4).

Case 2 The boundary condition $(\delta v,-\delta u,0)\cdot\boldsymbol{n}|_{\partial\Omega}=0$ means $(v,-u,0)\cdot\boldsymbol{n}$ is fixed on $\partial\Omega$. So we take $(v,-u,0)\cdot\boldsymbol{n}=(\tilde{v},-\tilde{u},0)\cdot\boldsymbol{n}$ where $(v,-u,0)\cdot\boldsymbol{n}|_{\partial\Omega}$ can be gained because of the known observational field $(\tilde{u},\tilde{v},\tilde{w})\cdot\zeta$ can be obtianed from Eq. (3.6), and then the analysis value (u,v,w) can be obtained by substituting (λ,ζ) into Eq. (3.4). Here the condition $(v,-u,0)\cdot\boldsymbol{n}=(\tilde{v},-\tilde{u},0)\cdot\boldsymbol{n}$ must be verified on $\partial\Omega$, otherwise the result of ζ will be wrong.

4. The analysis of numerical results

4.1 *Case 1* (*The first kind of boundary condition*)

The true flow field $u_{\rm t},v_{\rm t},w_{\rm t})$ is given by Eq. (2.5), and the observations $(\tilde{u},\tilde{v},\tilde{w})$, by Eq. (2.6). Our intention is to find an analysis value (u,v,w) to satisfy Eq. (2.1*) and make the cost function minimal, i.e., $\tilde{J}[u,v,w]$ =min. The domain of the search is $\Omega=(0,\pi)\times(0,\pi)\times(0,\pi)$ with the boundary $\partial\Omega=\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4\cup\Gamma_5\cup\Gamma_6$, just as shown in Fig. 1.

At first, we solve the following boundary value problem

$$\Delta\zeta-\frac{\zeta}{\gamma}=-\frac{\tilde{\zeta}}{\gamma}\,,\quad \zeta|_{\partial\Omega}=0\,.\qquad (4.1)$$

Using Eq. (2.6), Eq. (4.1) has the following form

$$\Delta\zeta-\frac{\zeta}{\gamma}=-\frac{\tilde{\zeta}}{\gamma}=-\frac{\varepsilon(n-m)\cos(nx)\cos(my)\sin(z)}{\gamma}\,.\qquad (4.1^*)$$

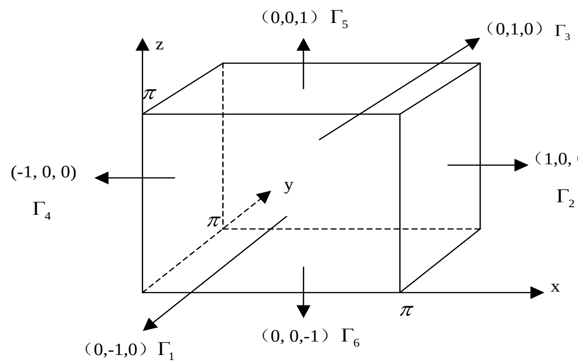


Fig. 1. The search domain.

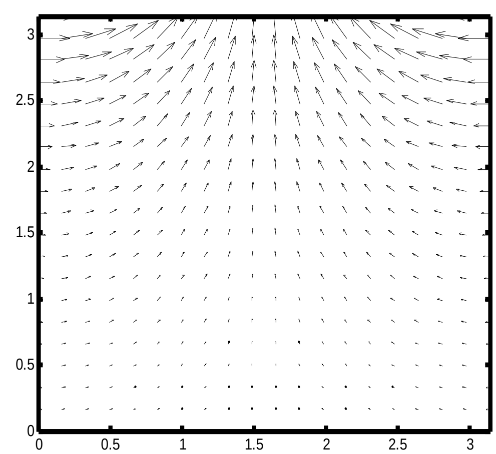


Fig. 2. The horizontal part in the $z=\pi/2$ plane for the true flow.

Expanding ζ into a series in terms of a infinite sequence of basis functions $\{\sin(a_0x)\sin(a_1y)\sin(a_2z)\}$

$$\zeta=\sum_{a_0,a_1,a_2=1}^{\infty}C_{a_0,a_1,a_2}\sin(a_0x)\sin(a_1y)\sin(a_2z)\qquad (4.2)$$

and substituting Eq. (4.2) into Eq. (4.1) yields

$$\sum_{a_0,a_1,a_2=1}^{\infty}-(a_0^2+a_1^2+a_2^2+\frac{1}{\gamma})\times C_{a_0,a_1,a_2}\sin(a_0x)\sin(a_1y)\sin(a_2z)=-\frac{\varepsilon(n-m)\cos(nx)\cos(my)\sin(z)}{\gamma}\,.\qquad (4.3)$$

Multiplying the two sides of (4.3) by $\sin(Px)\sin(Qy)\sin(Rz)$ and integrating over Ω leads

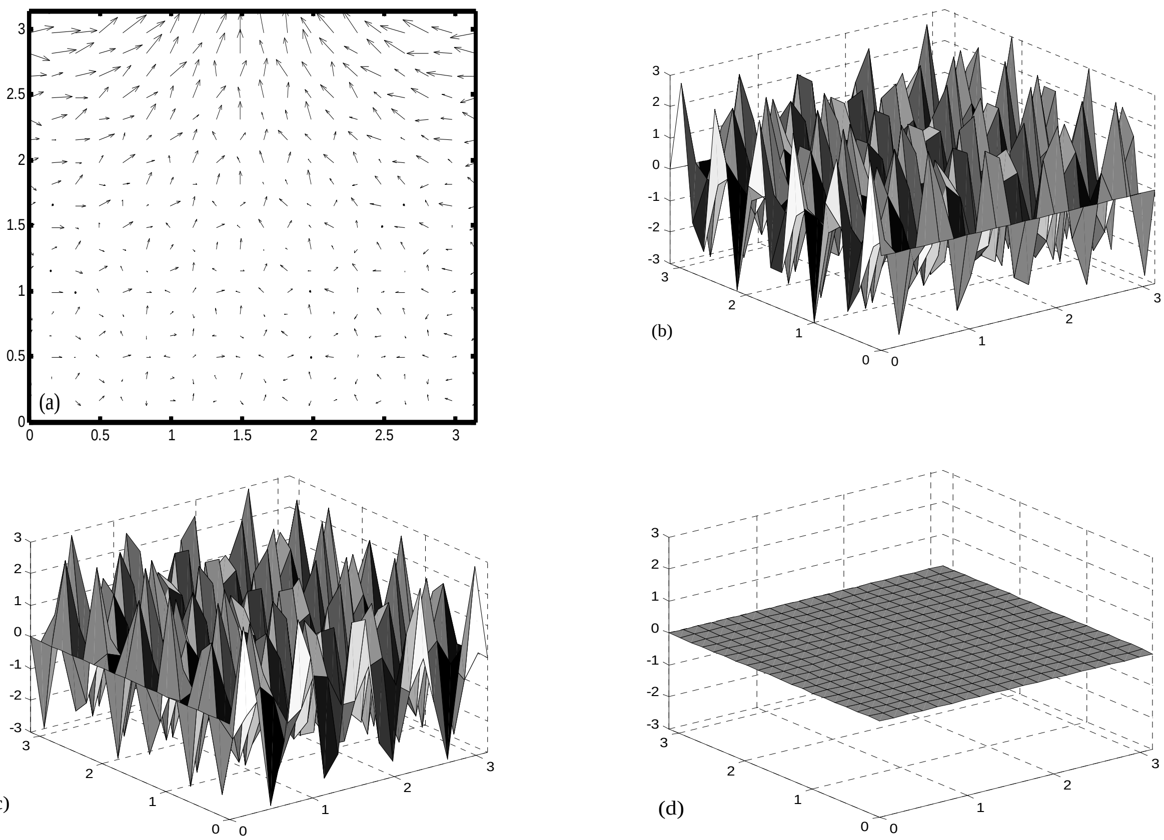


Fig. 3. (a) The horizontal wind field in the $z = \pi/2$ plane for the wind observations. (b) The discrepancy between the observations u and the true value u_t in the $z = \pi/2$ plane $\|u - u_t\|^2_{H_1} = 309.14\varepsilon^2$. (c) The difference between observations v and the true value u_t in the $z = \pi/2$ plane $\|v - v_t\|^2_{H_1} = 321.25\varepsilon^2$. (d) The discrepancy between observations w and the true value w_t in the $z = \pi/2$ plane $\|w - w\|^2_{H_1} = 0$.

to

$$C_{P,Q,R}\left(\frac{\pi}{2}\right)^3\left(P^2+Q^2+R^2+\frac{1}{\gamma}\right) = \frac{\varepsilon(n-m)}{\gamma}\int_0^\pi \sin(Px)\cos(nx)dx \times \int_0^\pi \sin(Qy)\cos(my)dy\int_0^\pi \sin(Rz)\sin(z)dz\,. \tag{4.4}$$

So

$$\zeta = \frac{64\varepsilon(n-m)}{\gamma\pi^2} \times \sum_{k_0,k=1}^{\infty}\frac{k_0k_1\sin(2k_0x)\sin(2k_1y)\sin(z)}{(4k_0^2-n^2)(4k_2^2-m^2)(4k_0^2+4k_1^2+1+\frac{1}{\gamma})} \tag{4.5}$$

Utilizing Eqs. (2.9), (3.4), and (4.5), the analyzed flow field (u, v, w) can be obtained as follows:

$$u = \tilde{u} - \frac{\varepsilon n(n+m)\cos(nx)\sin(my)\sin(z)}{n^2+m^2+1} - \frac{128\varepsilon(n-m)}{\pi^2} \times \sum_{k_0,k=1}^{\infty}\frac{k_0k_1^2\sin(2k_0x)\cos(2k_1y)\sin(z)}{(4k_0^2-n^2)(4k_2^2-m^2)(4k_0^2+4k_1^2+1+\frac{1}{\gamma})}\,, \tag{4.6a}$$

$$v = \tilde{v} - \frac{\varepsilon m(n+m)\sin(nx)\cos(my)\sin(z)}{n^2+m^2+1} + \frac{128\varepsilon(n-m)}{\pi^2} \times \sum_{k_0,k=1}^{\infty}\frac{k_0^2k_1\cos(2k_0x)\sin(2k_1y)\sin(z)}{(4k_0^2-n^2)(4k_2^2-m^2)(4k_0^2+4k_1^2+1+\frac{1}{\gamma})}\,, \tag{4.6b}$$

$$w = \tilde{w} - \frac{\varepsilon(n+m)\sin(nx)\sin(my)\cos(z)}{n^2+m^2+1}\,. \tag{4.6c}$$

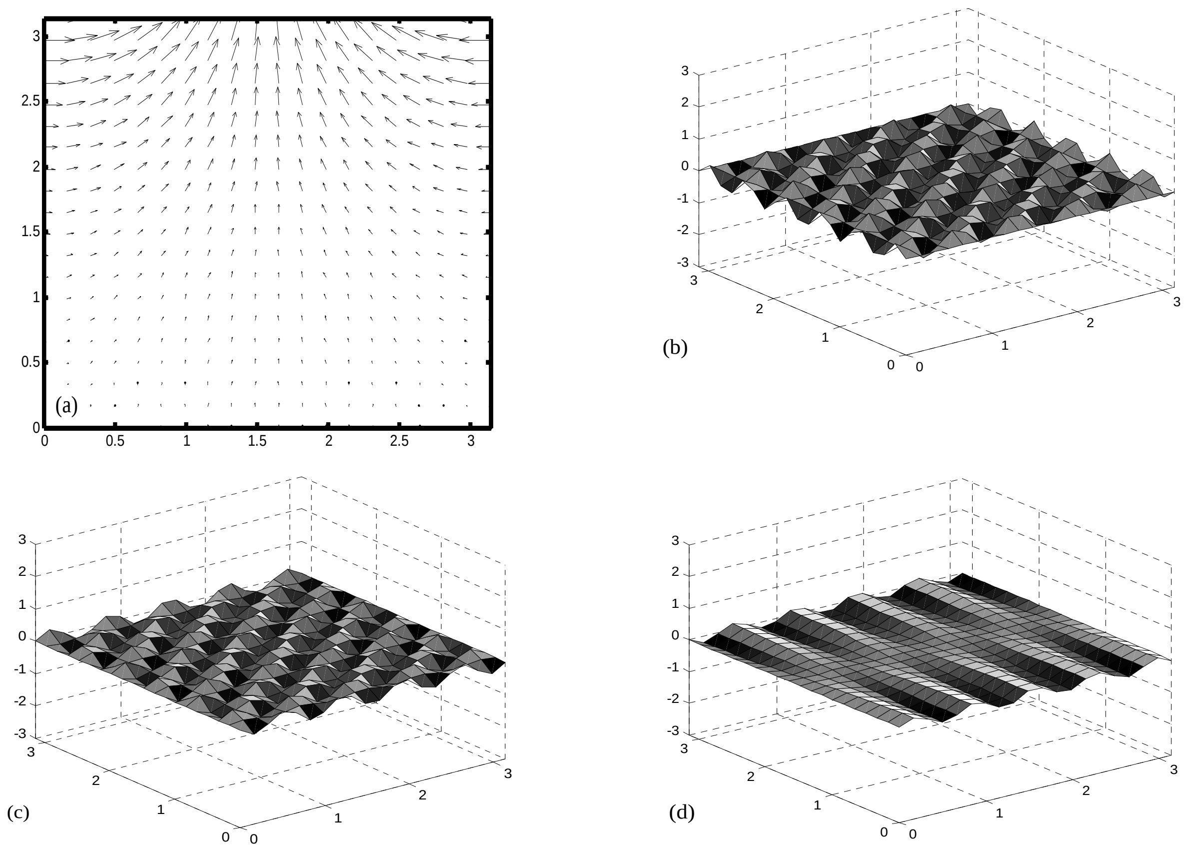


Fig. 4. (a) The analyzed horizontal flow field (u, v) using MVA. (b) The discrepancy between the true value $u_{\rm t}$ and the analyzed value u using MVA $\|u - u_{\rm t}\|^2_{H_1} = 3.96\varepsilon^2$. (c) The discrepancy between the true value $v_{\rm t}$ and the analyzed value v using MVA $\|v - v_{\rm t}\|^2_{H_1} = 3.21\varepsilon^2$. (d) The discrepancy between the true value $w_{\rm t}$ and the analyzed value w using MVA $\|w - w_{\rm t}\|^2_{H_1} = 0.92\varepsilon^2$.

4.2 *Case 2* (*The second kind of boundary condition*)

In this case, the boundary condition is $(v, -u, 0) \cdot \boldsymbol{n} = (\tilde{v}, -\tilde{u}, 0) \cdot \boldsymbol{n}$. So

$$\left\{\begin{array}{l}
(\tilde{v}, -\tilde{u}, 0) \cdot \boldsymbol{n}|_{\Gamma 1} = \tilde{u}|_{y=0} = 0; \\
(\tilde{v}, -\tilde{u}, 0) \cdot \boldsymbol{n}|_{\Gamma 2} = \tilde{v}|_{x=\pi} = 0; \\
(\tilde{v}, -\tilde{u}, 0) \cdot \boldsymbol{n}|_{\Gamma 3} = -\tilde{u}|_{y=\pi} = -\cos(x)\mathrm{sh}(\pi)\sin(z); \\
(\tilde{v}, -\tilde{u}, 0) \cdot \boldsymbol{n}|_{\Gamma 4} = \tilde{v}|_{x=0} = 0; \\
(\tilde{v}, -\tilde{u}, 0) \cdot \boldsymbol{n}|_{\Gamma 3} = 0|_{z=\pi} = 0; \\
(\tilde{v}, -\tilde{u}, 0) \cdot \boldsymbol{n}|_{\Gamma 6} = 0|_{z=0} = 0\,.
\end{array}\right. \tag{4.7}$$

The equation of ζ is Eq. (4.1*). We apply the undetermined coefficients method to solve Eqs. (4.1*) and (4.7). Supposing

$$\zeta = A\cos(nx)\cos(my)\sin(z)\,,$$

where A is a constant and will be determined. Substituting ζ into Eq. (4.1*) gives

$$\zeta = \frac{\varepsilon(n-m)}{\gamma(n^2+m^2+1)+1}\cos(nx)\cos(my)\sin(z)\,. \tag{4.8}$$

The analyzed field is (u, v, w) is

$$u = \tilde{u} - \left[\frac{n(n+m)}{n^2+m^2+1} - \frac{\gamma m(n-m)}{\gamma(n^2+m^2+1)+1}\right] \times \varepsilon\cos(nx)\sin(my)\sin(z)\,, \tag{4.9a}$$

$$v = \tilde{v} - \left[\frac{n(n+m)}{n^2+m^2+1} - \frac{\gamma n(n-m)}{\gamma(n^2+m^2+1)+1}\right] \times \varepsilon\sin(nx)\cos(my)\sin(z)\,, \tag{4.9b}$$

$$w = \tilde{w} - \frac{\varepsilon(n+m)\sin(nx)\sin(my)\cos(z)}{n^2+m^2+1}\,, \tag{4.9c}$$

The result of ζ is consistent with (4.7), so the result is proper.

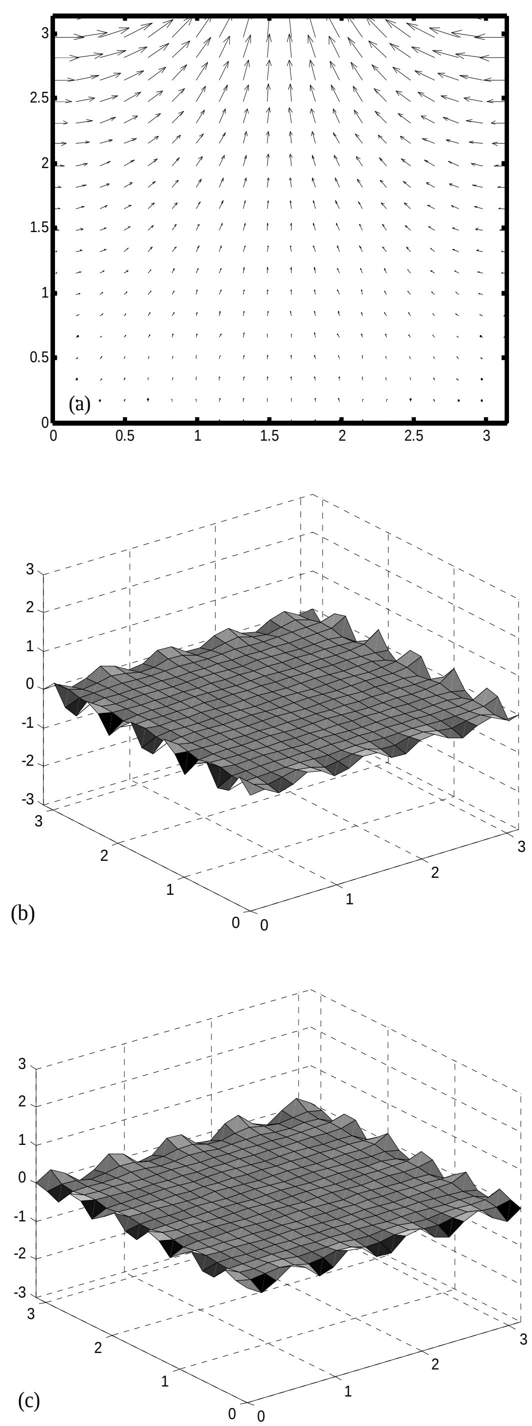


Fig. 5. (a) The analyzed horizontal flow field (u, v) of case 1 using the GMVA. (b) The discrepancy between the true value u_{t} and the analyzed value u of case 1 using the GMVA. $\|u - u_{\mathrm{t}}\|^2_{H_1} = 0.76\varepsilon^2$. (c) The discrepancy between value v_{t} and the analyzed value v of case 1 using the GMVA $\|v - v_{\mathrm{t}}\|^2_{H_1} = 0.59\varepsilon^2$.

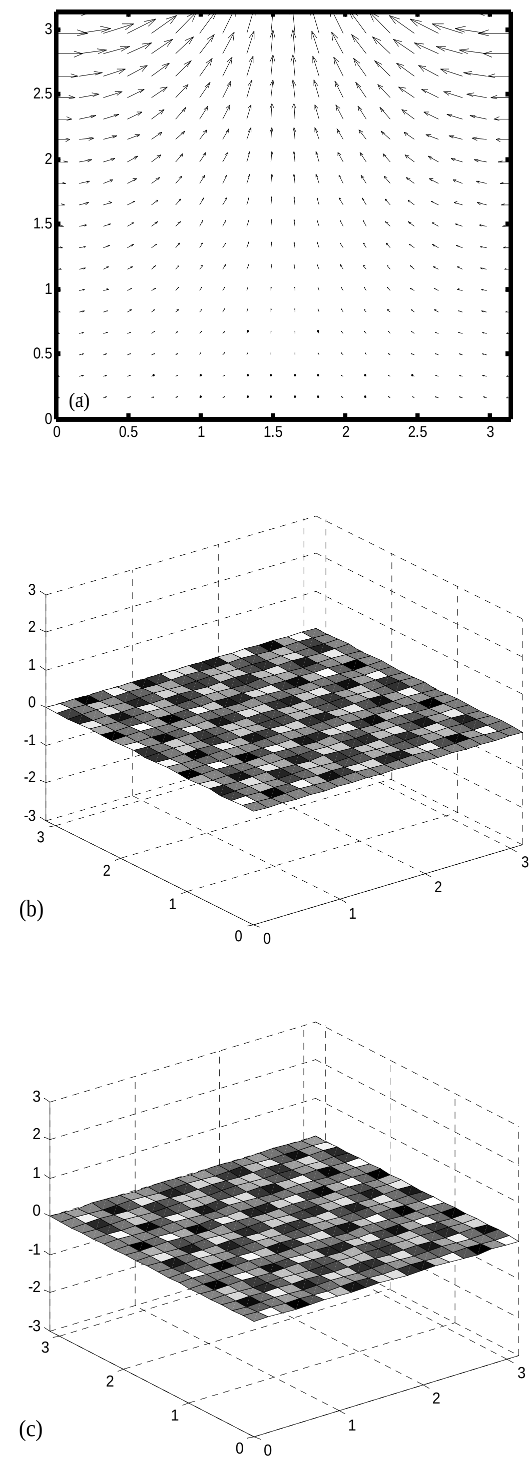


Fig. 6. (a) The analyzed horizontal flow field (u, v) of case 2 using the GMVA. (b) The discrepancy between true value u_{t} and the analyzed value u of case 2 using the GMVA. $\|u - u_{\mathrm{t}}\|^2_{H_1} = 0.074\varepsilon^2$. (c) The discrepancy between true value v_{t} and the analyzed value v of case 2 using the GMVA $\|v - v_{\mathrm{t}}\|^2_{H_1} = 0.074\varepsilon^2$.

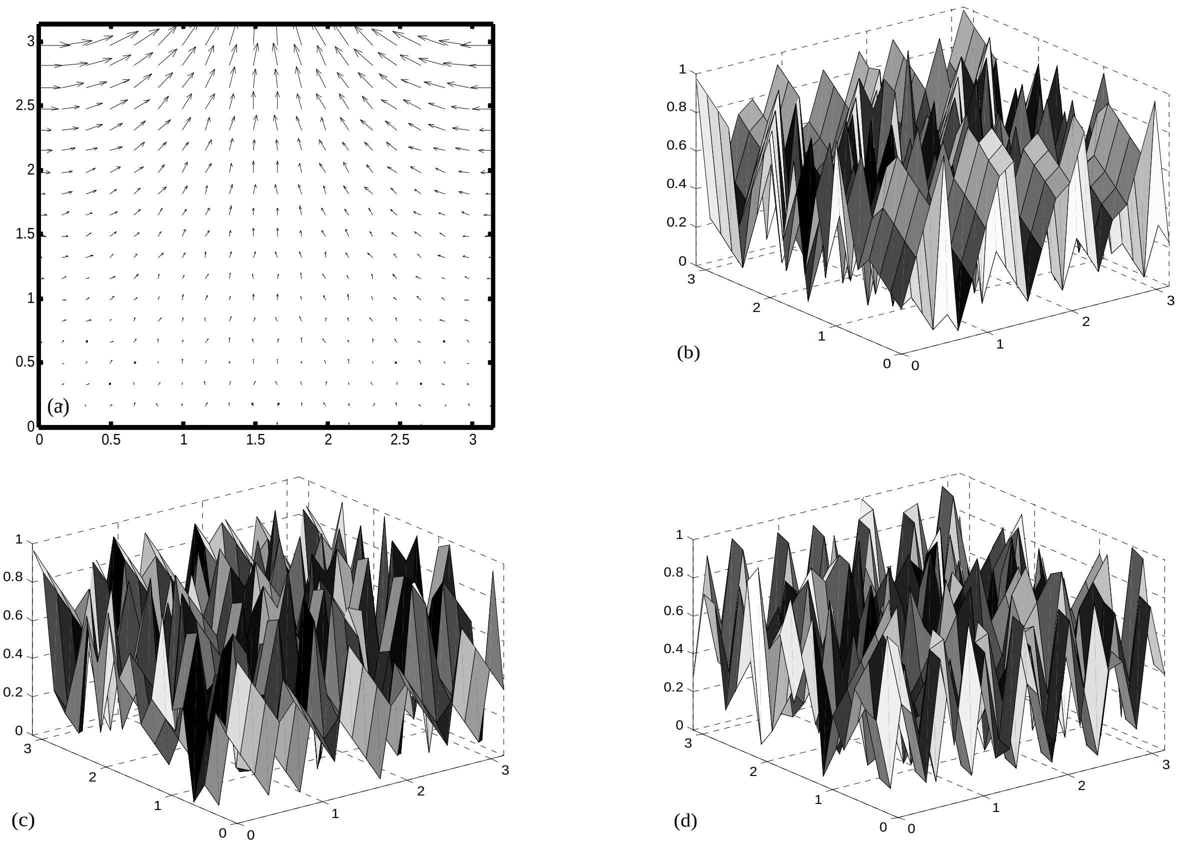


Fig. 7. (a) The horizontal flow field with a random disturbance in the $z = \pi/2$ plane. (b) The discrepancy between the true value u_t and the observations u containing a random disturbance $\|u - u_t\|^2_{H_1} = 321.58\varepsilon^2$. (c) The discrepancy between the true value v_t and the observations v containing a random disturbance $\|v - v_t\|^2_{H_1} = 313.81\varepsilon^2$. (d) The discrepancy between true value w_t and the observations w containing a random disturbance $\|w - w_t\|^2_{H_1} = 318.83\varepsilon^2$.

4.3 *The adjustment of observational wind with random disturbance*

The true flow field in this section is still Eq. (2.5), and the wind observations are defined as the true flow field plus a random disturbance, namely

$$\begin{cases} \tilde{u} = u_t + u_{\text{random}} \ , \\ \tilde{v} = v_t + v_{\text{random}} \ , \\ \tilde{w} = w_t + w_{\text{random}} \ . \end{cases} \tag{4.10}$$

Where urandom, vrandom, and wrandom are random numbers in the interval (0,1). The stable cost function is still Eq. (3.1), and the boundary condition is the same as Eq. (4.1). As discussed above, we still have Eqs. (4.1) and (3.6).

The random disturbances urandom, vrandom, and wrandom must be filtered because they contain high frequency component (Zeng, 1997). The method is as follows.

Let ϕ and $\overline{\phi}$ stand for the unfiltered and filtered data respectively. Then $\sum\limits_{i,j,k}(\phi_{i,j,k} - \overline{\phi}_{i,j,k})^2$ is required to reach its minimum under the constraint

$$\begin{aligned} \overline{\phi}_{i,j,k} - \overline{\phi}_{i,j-1,k} &\cong 0 \ , \\ \overline{\phi}_{i,j,k} - \overline{\phi}_{i-1,j,k} &\cong 0 \ , \\ \overline{\phi}_{i,j,k} - \overline{\phi}_{i,j,k-1} &\cong 0 \ , \end{aligned}$$

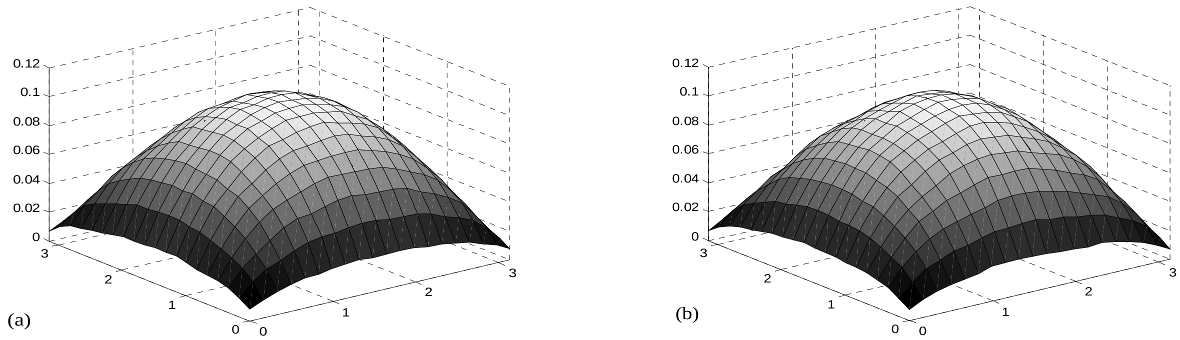
where $i = 1, 2, \ldots, n_0, j = 1, 2, \ldots, n$, and $k = 1, 2, \ldots, n_2$, and n_0, n_1, and n_2 are the dimensions of three directions respectively.

The cost function can be written as

$$\begin{aligned} J_1 = \sum_{i,j,k} \{ (\phi_{i,j,k} - \overline{\phi}_{i,j,k})^2 + \mu[(\overline{\phi}_{i,j,k} - \overline{\phi}_{i-1,j,k})^2 \\ + (\overline{\phi}_{i,j,k} - \overline{\phi}_{i,j-1,k})^2 + (\overline{\phi}_{i,j,k} - \overline{\phi}_{i,j,k-1})^2] \} = \min \ . \end{aligned}$$

According to the necessary condition for J_1 to reach a minimum, it follows that

$$\begin{aligned} \phi_{i,j,k} = &\overline{\phi}_{i,j,k} - \mu(\overline{\phi}_{i+1,j,k} + \overline{\phi}_{i-1,j,k} + \overline{\phi}_{i,j+1,k} \\ &+ \overline{\phi}_{i,j-1,k} + \overline{\phi}_{i,j,k+1} + \overline{\phi}_{i,j,k-1} - 6\overline{\phi}_{i,j,k}) \ . \end{aligned} \tag{4.11}$$



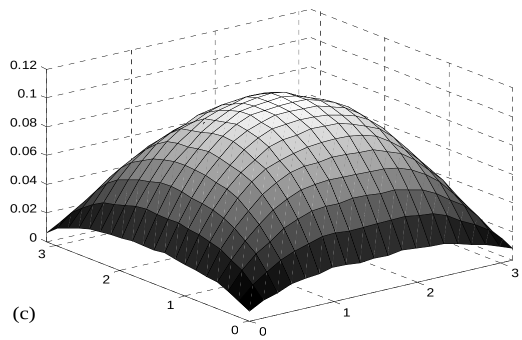


Fig. 8. (a) The discrepancy of u by low-pass filter adjustment $\|u - u_{\mathrm{t}}\|_{H_1}^2 = 2.47\varepsilon^2$. (b) The discrepancy of v by low-pass filter adjustment $\|v - v^T\|_{H_1}^2 = 2.36\varepsilon^2$. (c) The discrepancy of w by low-pass filter adjustment $\|w - w^T\|_{H_1}^2 = 2.43\varepsilon^2$.

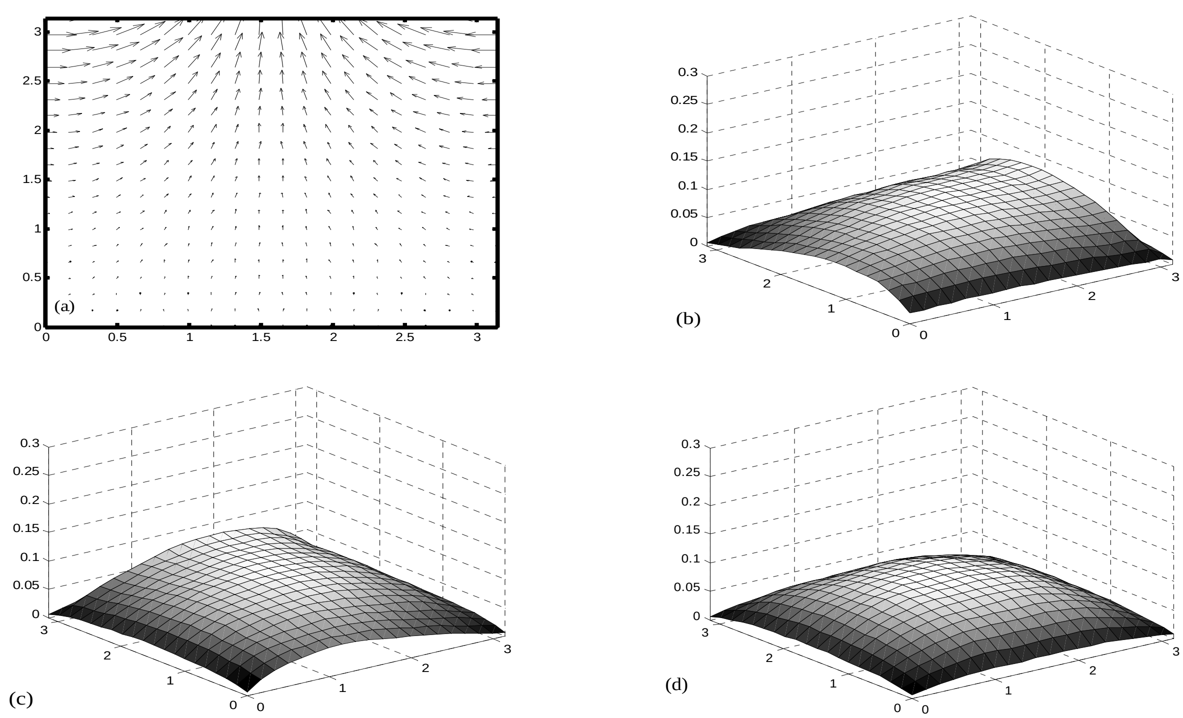


Fig. 9. (a) The analyzed horizontal flow field (u, v) using MVA for the observations containing a random disturbance. (b) The discrepancy between the true value u_{t} and the analyzed value u using MVA $\|u - u_{\mathrm{t}}\|_{H_1}^2 = 2.16\varepsilon^2$. (c) The discrepancy between true value v_{t} and the analyzed value v using MVA $\|v - v_{\mathrm{t}}\|_{H_1}^2 = 2.16\varepsilon^2$. (d) The discrepancy between the true value w_{t} and the analyzed w using MVA $\|w - w_{\mathrm{t}}\|_{H_1}^2 = 2.15\varepsilon^2$.

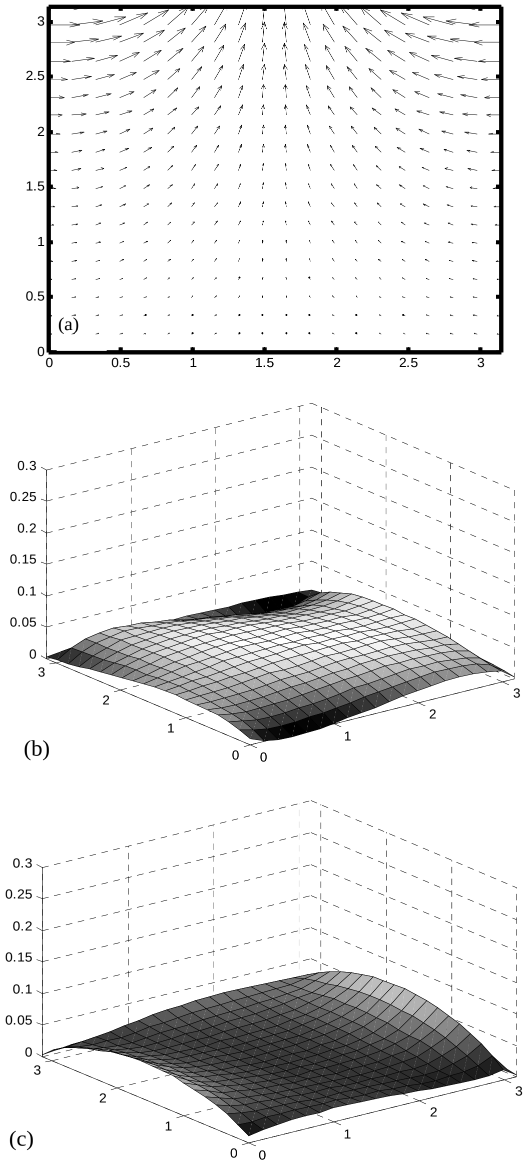


Fig. 10. (a) The analyzed horizontal flow field (u, v) containing a random disturbance using the GMVA. (b) The discrepancy between the true value u_{t} and the analyzed value u using the GMVA $\|u - u_{\mathrm{t}}\|^2_{H_1} = 1.04\varepsilon^2$. (c) The discrepancy between the true value v_{t} and the analyzed value v using the GMVA $\|v - v_{\mathrm{t}}\|^2_{H_1} = 1.15\varepsilon^2$.

Here μ is called the filtering coefficient, and the larger μ is, the more serious the short wave attenuation. In this section, we take $\mu = 100$, $\gamma = 5$.

Using filtered $u_{\mathrm{random}}, v_{\mathrm{random}}, w_{\mathrm{random}}$, we repeat the above procedures to obtain an analyzed field, which is shown in Figs. 7–10.

5. Concluding remarks

In this paper, the generalized method of variational analysis (GMVA) for 2-D wind observations developed by Huang et al. (2004) is extended to 3-D cases, which can deal well with wind fields containing high frequency components.

For the true and observational flow fields given by Eqs. (2.5) and (2.6) respectively, and the constant vertical component of the true wind field, we first calculate the analyzed value of the observational wind field using MVA, and then calculate it using the GMVA under two kinds of boundary conditions. The results are illustrated in Figs. 1–6. The efficiency of the GMVA used in the present paper is compared to that of the MVA by Sasaki. In the end, a filtering technique is introduced to cope with the observational wind containing random disturbances, and the results are just as indicated in Figs. 7–10. The following conclusions can be made.

(1) When the observations contain high frequency noise, the discrepancy of w can be reduced to some extent while the discrepancy of u, v is not small, even after filtering.

(2) The analyzed flow field calculated by the GMVA in case 1 is approximately equal to the true field when the series are truncated at $k_0 = 20, k_1 = 20$. In particular when $x \in (0.1, 3.0)$, the regulated flow field is almost equal to the true flow field. But at $x = 0, x = \pi$, there are some discrepancies, which can be improved by increasing the rank of the truncation.

(3) The analyzed flow field using the GMVA in case 2 is almost equal to the true field.

(4) For the case of the observational field containing random disturbances, filtering is applied first, and then an accurate analyzed flow field can be obtained using the GMVA.

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