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Simple General Atmospheric Circulation and Climate Models with Memory

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ABSTRACT

This article examines some general atmospheric circulation and climate models in the context of the notion of "memory". Two kinds of memories are defined: statistical memory and deterministic memory. The former is defined through the autocorrelation characteristic of the process if it is random (chaotic), while for the latter, a special memory function is introduced. Three of the numerous existing models are selected as examples. For each of the models, asymptotic (at $t \to \infty$) expressions are derived. In this way, the transients are filtered out and that which remains concerns the final behaviour of the models.

Key words: atmospheric circulation, climate, memory, model

1. Introduction

The notion of "memory" is an interdisciplinary one. One talks of human, animal and even plant memory in nature, of machine (computer) memory in electronics, and so on. Memory is a kind of information storage for the past events to be used for current or future needs. In quantifying this phenomenon, various measures have been introduced. In this connection, we distinguish two kinds of memories: statistical memory (SM) and deterministic memory (DM).

(1) Statistical memory. Let $X(t)$ be a random stationary function of time t with a zero mean value $\langle X\rangle$=0 and an autocorrelation function (ACF):

$$B(\tau) = \langle X(t)X(t-\tau)\rangle \ . \tag{1a}$$

Then $B(0) = \langle X^2\rangle = \sigma_x^2$ is the variance, while

$$R(\tau) = B(\tau)/B(0) \tag{1b}$$

is the autocorrelation coefficient (ACC). Obviously, $R(\tau) = R(-\tau)$ and $R(0)$=1. Normally $R(\tau) \to 0$ at $\tau \to \infty$. The ACC is a measure of the statistical correlation between two values of $X(t)$ separated by the time interval τ. In essence, this is SM. Now a question arises about the typical timescale $\tau_{\rm s}$ (duration) of SM. By definition

$$\tau_{\rm s} = \int_0^\infty R(\tau)d\tau \ . \tag{2}$$

When the integral is finite, there exists a typical time memory, also called correlation time (Panchev, 1971). For example, if $R(\tau) = \exp(-\tau/\tau_c)$ with a finite ($<\infty$) and small τ_c, then $\tau_c \equiv \tau_{\rm s}$ (see also the Appendix). Such a process is said to be short-range correlated. However, if τ_c is very large ($\tau_c \to \infty$ and $R(\tau)$ decays very slowly), or $R(\tau) \sim \tau^{m-1}, 0 < m \leqslant 1$, then $\tau_{\rm s} = \infty$ and the process is said to be long-range correlated. In this case, it is impossible to select a timescale separating the regime of temporal correlation from that of pairwise independence.

(2) Deterministic memory. Let us consider the ordinary differential equation

$$\dot{M} + kM = S(t) \ , \quad k > 0 \tag{3}$$

with $\dot{M} = dM/dt$, initial condition $M(0) = M_0$ and

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k =constant>0. Its general solution is

$$M(t) - M_0 e^{-kt} = \int_0^t e^{-k(t-u)} S(u) du = \int_0^t e^{-k\tau} S(t-\tau) d\tau \,. \quad (4)$$

Hence, at $t \to \infty$

$$M(t \to \infty) = M_\infty(t) = \int_0^\infty e^{-k\tau} S(t-\tau) d\tau \,. \quad (5)$$

Therefore, the current value $M(t)$ is determined not only by $S(t)$, but by $S(t-\tau)$ as well, the latter being exponentially damped. Because of the delayed argument $(t-\tau)$, it is reasonable to interpret the integral term in Eqs. (4) and (5) as a memory, remaining after the transient has passed ($t \to \infty$). To characterize the latter effect, we introduce the memory function

$$\tilde{S}_{k,n}(t) = k \int_0^\infty e^{-k\tau} S^n(t-\tau) d\tau \quad (6)$$

with $n = 1$ (linear DM), or $n = 2$ (quadratic one) in the applications here. Obviously

$$\tilde{S}_{k,1}(t) = k M_\infty(t) \,, \;\; k \int_0^\infty e^{-k\tau} d\tau = 1 \,. \quad (7)$$

In the special case of very large values of k (i.e., very short memory), one can approximate $e^{-k\tau}$ by a delta function $\delta(k\tau)$ and then perform the integration in (6):

$$\tilde{S}_{k,n}(t) \approx S^n(t) \,. \quad (8)$$

For example, let $S(t) = \sin t$. Then

$$\tilde{S}_{k,1}(t) = \frac{k}{1+k^2}(k \sin t - \cos t) \xrightarrow[k \to \infty]{} \sin t$$
$$\tilde{S}_{k,2}(t) = \frac{1}{2} - \frac{k}{8+2k^2}(k \cos 2t + 2 \sin t) \xrightarrow[k \to \infty]{} \sin^2 t \,.$$

Therefore, Eq. (5)–(8) yield

$$M_\infty(t) \approx \frac{1}{k} S(t) \,. \quad (9)$$

However, this result is equivalent to that following from Eq. (3) at $\dot{M} = 0$ ($kM \approx S$), and $M(t)$ determined from here has to be identified as $M_\infty(t)$, i.e., valid at $t \to \infty$.

So far, nothing has been said about the function $S(t)$ in Eq. (3). If it is a regular and integrable one, then $M(t)$ is fully determined by Eqs. (4) and (5). As to the typical time memory scale τ_d in (6), it obviously equals

$$\tau_d \sim 1/k \,. \quad (10)$$

At large k, the exponent tends to zero very quickly (short deterministic memory). However, if $S(t)$ is a random function [stochastic or chaotic (Panchev, 1971)], then formally Eq. (5) remains in force and $M_\infty(t)$ is a random function too, defined by the integral transformation. Therefore, it can be treated statistically. In particular, an integral expression for the ACF of $\tilde{S}_{k,1}(t)$ can be derived (Panchev, 1971)

$$\tilde{B}(\tau) = k \int_0^\infty e^{-ku} B(\tau + u) du \,, \quad (11)$$

and the time scale $\tilde{\tau}_s(k, \tau_s)$ corresponding to Eq. (2) can be calculated (see the Appendix for examples). In what follows, we are going to consider some selected General Atmospheric Circulation (GAC) and Climate Models (CMs) from the above described positions.

2. The GAC and CMs

The GAC and climate modeling both are interrelated problems of the present studies of the long range behaviour of the Earth's atmosphere. A great number of models have been developed. They can be classified into two main groups:

Group A: The global atmosphere or atmosphere-ocean coupled mathematical models in the form of nonlinear systems of partial differential equations (PDEs), capable of simulating satisfactory global and regional phenomena. The present computer capabilities permit direct numerical simulation and experiments based on these equations. Nevertheless, the spectral (Galerkin type) decomposition with subsequent truncation remains a widely used method for treating the original PDEs and their reduction to low-order nonlinear dynamical systems (NDSs)—sets of first-order ordinary differential equations (ODEs)—(Panchev, 2001). Two examples are considered below.

Group B: Directly-constructed low-order NDSs for some globally-averaged climatic characteristics (temperature, ice cover, CO_2 concentration, etc.). One example is considered below.

Both approaches are currently followed, depending on the purpose of the research. Some of the models allow an effective application of analytical tools as well.

Formally, the situation is quite similar to that in Earth magnetic field (geodynamo) theory. The same two groups of models can be identified (Panchev, 2001): Global models "producing" low-order NDSs (models) and directly-built-up, exact, low-order models for laboratory devices, simulating some features of the natural earth magnetic field (e.g., polarity reversals). Because of this, a transfer of ideas from one problem to the other is possible.

We now consider some examples, all in dimensionless form.

2.1 *Example 1. The Lorenz GAC model* (*Lorenz, 1984, 1990*)

The model's equations are:

$$\begin{cases} \dot{X} = -Y^2 - Z^2 - aX + aF \,, \\ \dot{Y} = XY - bXZ - Y + G \,, \\ \dot{Z} = bXY + XZ - Z \,, \end{cases} \tag{12}$$

where $X(t)$ represents the westerly wind current (equivalently the poleward temperature gradient), while $Y(t)$ and $Z(t)$ represent the amplitudes of the cosine and sine phases of the strength of the large scale eddies in the atmosphere. The terms aF and G stand for the symmetric and asymmetric external thermal forcing due to astro-geo-factors and land-ocean distribution on the globe, respectively. Finally, $a < 1$ and $b > 1$ are parameters of the model. Regardless of its simplicity, Eq. (12) is a powerful model, capable of simulating many features of the real midlatitude atmosphere (Roeber, 1995; Roeber et al., 1997; Gonzalez-Miranda, 1997; Masoller et al., 1992).

Generally, $F = F(t)$ and $G = G(t)$ mainly because of the interannual (seasonal) variability of the insolation. By introducing new variables

$$U(t) = Y^2(t) + Z^2(t) \,, \;\; V(t) = X(t) - F(t) \tag{13}$$

the original system (12) can be written in the form

$$\begin{cases} \dot{V} + aV = -U(t) - \dot{F}(t) \,, \\ \dot{U} = 2U[V + F(t) - 1] + 2G(t)Y \,. \end{cases} \tag{14}$$

The first equation is identical to Eq. (3), and according to Eq. (5)

$$V_\infty(t) = -\frac{1}{a}[\tilde{U}_{a,1}(t) + \tilde{\dot{F}}_{a,1}(t)] \,. \tag{15}$$

However

$$\tilde{\dot{F}}_{a,1} = \int_0^\infty e^{-a\tau}\dot{F}(t-\tau)d\tau = a[F(t) - \tilde{F}_{a,1}(t)] \,. \tag{16}$$

For further use of the original system (12) or its alternative (14), one has to specify the driving parameters $F(t)$ and $G(t)$. Beginning with Lorenz (1990), two particular cases have been most often examined:

1. F=constant, and G=constant which corresponds to perpetual winter (summer) conditions;
2. $F(t) = F_0 + F_1 \sin\omega t$, $G(t)$=constant, $\omega = 2\pi/T$, $T = 1$ year, which corresponds to the presence of seasons.

With fixed F=constant and G=0 (homogeneous earth surface, e.g., no continents), the system (14) becomes an autonomous two-dimensional one, and consequently it cannot generate chaotic solutions. Thus, the final behaviour will be governed by the following integro-differential equation with delaying argument

$$\dot{U}_\infty = 2U_\infty \left[F - 1 - \int_0^\infty e^{-a\tau}U_\infty(t-\tau)d\tau\right] \,, \tag{17}$$

and by Eq. (15) with $\dot{F} = 0$. This pair of equations has two stationary solutions (fixed points)—a trivial one $\bar{U} = \bar{V} = 0$ and a nontrivial one $\bar{U} = a(F-1)$,$\bar{V} = 1 - F$, interpreted by Lorenz (1984) and by other authors cited above. As to the transient process (0< $t < \infty$), it is governed either by Eq. (14) or by the equivalent equation of known (Lienard) type

$$\ddot{V} + [a - 2(F-1) - 2V]\dot{V} - 2a(F - 1 + V)V = 0 \,. \tag{18}$$

Under suitable parameter values, it allows limit-cycle solutions (Perko, 1996). Moreover, with nonzero constants F and G, a full energy ($E = X^2 + Y^2 + Z^2$) equation follows from Eq. (12)

$$\dot{E} + 2E = 2(1-a)X^2 + 2aFX + 2GY \,.$$

Hence, asymptotically

$$E_\infty(t) = (1-a)\tilde{X}_{2,2}(t) + aF\tilde{X}_{2,1}(t) + G\tilde{Y}_{2,1}(t) \,.$$

Much more realistic is the second assumption of sinusoidally-varying $F(t)$ and fixed G=0 or $G \neq$0. In this case, the general expression (16) yields

$$\tilde{\dot{F}}_{a,1}(t) = \frac{a\omega F_1}{\omega^2 + a^2}(\omega \sin\omega t + a\cos\omega t) \,,$$

to be introduced into (15).

2.2 *Example 2. The Palmer-Lorenz (P-L) model*

It is a well known fact in meteorology that the extratropical atmosphere behaves much more irregularly (chaotically) than the tropical one. To study qualitatively this peculiarity, Palmer (1993, 1995) extended the classical Lorenz (1963) low-order model by adding new terms in the equations, accounting for the storage effect of the sea surface temperature anomalies. Thus, the extended model reads

$$\begin{cases} \dot{X} = -\sigma X + \sigma Y + M(t) \,, \\ \dot{Y} = -XZ + rX - Y + N(t) \,, \\ \dot{Z} = XY - bZ \,. \end{cases} \tag{19}$$

Palmer assumed that

$$\begin{cases} M(t) = \alpha \int_{-\infty}^t X(t')e^{-c(t-t')}dt' \,, \\ N(t) = \alpha \int_{-\infty}^t Y(t')e^{-c(t-t')}dt' \,, \end{cases} \tag{20}$$

which is the same as

$$\begin{cases} M(t) = \alpha \int_0^\infty e^{-c\tau}X(t-\tau)d\tau = \dfrac{\alpha}{c}\tilde{X}_{c,1}(t) \,, \\ N(t) = \alpha \int_0^\infty e^{-c\tau}Y(t-\tau)d\tau = \dfrac{\alpha}{c}\tilde{Y}_{c,1}(t) \,. \end{cases} \tag{21}$$

Obviously the additive terms $M(t)$ and $N(t)$ describe an artificially introduced memory in the classical Lorenz model (with $M = N = 0$) and the memory decay time equals $1/c$. However, the expressions (21) represent asymptotic (at $t \to \infty$) solutions of the linear equations

$$\dot{M} + cM = \alpha X(t) \ , \ \ \dot{N} + cN = \alpha Y(t) \ , \qquad (22)$$

so that Palmer's modification (19) and (20) is equivalent to the coupled five-dimensional system (19) and (22), valid however at $t \to \infty$, i.e., far from the initial disturbances.

On the other hand, the original Lorenz system ($M = N = 0$) has its own (endogenous) memory (Panchev and Spassova, 2004). Actually, a combination between the X- and Z-equations in Eq. (19) with the subsequent use of Eqs. (3) and (5) yields

$$Z_\infty(t) = \frac{1}{2\sigma} X^2(t) + \left(1 - \frac{b}{2\sigma}\right) \frac{1}{b} \tilde{X}_{b,2}(t) + \frac{\alpha}{bc} \tilde{P}_{b,1}(t) \ , \qquad (23)$$

where $P(t) = X(t)M(t)$ and

$$\tilde{P}_{b,1}(t) = b \int_0^\infty e^{-b\tau'} P(t - \tau') d\tau' = b \int_0^\infty \int_0^\infty e^{-c\tau - b\tau'} X(t - \tau') X(t - \tau - \tau') d\tau d\tau' \ .$$

Obviously, $\tilde{P}_{b,1}(t)$ is a more complicated quadratic (with respect to X) type of deterministic memory. The middle term in Eq. (23) is just the memory inherent to the original Lorenz model. Palmer's results indicate that the model (19) and (20) based on a simple paradigm for the chaotic extratropics is a good tool. Even invariant forcing $M = N =$constant in Eq. (19) still works well (Palmer, 1995). Here, we refined the model validity.

2.3 *Example 3. The Saltzman-Maasch (S-M) model*

Unlike the previous two models, this one (Saltzman and Maasch, 1988; Maasch and Saltzman, 1990) belongs entirely to group B. It concerns the late Pleistocene ice age oscillation and consists of three ODEs

$$\begin{cases} \dot{X} = -X - Y \ , \\ \dot{Y} = -pZ + rY - sZ^2 - Z^2 Y \ , \\ \dot{Z} = -q(X + Z) \ , \end{cases} \qquad (24)$$

where $X(t)$, $Y(t)$ and $Z(t)$ stand for the global ice mass, the CO_2 content and the mean static stability of the world ocean (i.e., the Brunt-Vaisala frequency) respectively, while p, q, r, s are model parameters. The Z-equation is of the form (3) so that

$$Z_\infty(t) = -q \int_0^\infty e^{-q\tau} X(t - \tau) d\tau = -\tilde{X}_{q,1}(t) \ . \qquad (25)$$

Then, the X- and Y-equations yield to a single second order (in essence nonlinear integro-differential) equation

$$\ddot{X}_\infty + (1 - rZ_\infty^2)\dot{X}_\infty - (r - Z_\infty^2)X_\infty = pZ_\infty + sZ_\infty^2 \ , \qquad (26)$$

where $Z_\infty(t)$ is given by (25). Therefore, asymptotically (at $t \to \infty$) the S-M model (24) behaves as a multiple self-forced harmonic oscillator (26). In the case of very short memory ($q \gg 1$), according to (8), Equation (26) becomes purely differential ($z_\infty(t) \approx -X_\infty(t)$).

The list of model examples can be continued.

3. Summary and conclusion

The atmosphere is a continuous medium with temporally-variable, interrelated phenomena and processes. Normally, the current values of the state characteristics depend on the previous ones along some typical time interval T. In other words, there exists some kind of memory for the past of the process. We distinguished two kinds of memory: statistical memory and deterministic memory. The former was defined through the autocorrelation characteristics (1) and (2) (see also (A.1)–(A.6)), while for the latter, a special memory function (6) was introduced, based on the asymptotic solution of Eq. (3).

In section 2, we analyzed three representative general atmospheric circulation and climate models for which equations of type (3) were identified. This allowed isolation of the final behaviour (at $t \to \infty$) from the transient one (at $t_0 < t < \infty$).

This approach can be applied to any other dynamical system containing an equation of type (3), or such an equation can be derived from the original equations of the system.

APPENDIX

Two examples for the memory characteristics

1. Let $X(t)$ be a random stationary process with an exponentially decaying autocorrelation function

$$B(\tau) = \sigma^2 e^{-c\tau} \ . \qquad (A.1)$$

Obviously

$$\tau_s = \int_0^\infty e^{-c\tau} d\tau = 1/c \ , \qquad (A.2)$$

while

$$\tilde{B}(\tau) = k \int_0^\infty e^{-ku} B(\tau + u) du = \frac{k}{k + c} B(\tau) \ . \qquad (A.3)$$

Therefore

$$\tilde{B}(\tau) < B(\tau) \text{ but } \tilde{R}(\tau) \equiv R(\tau) \text{ and } \tilde{\tau}_{\rm s} \equiv \tau_{\rm s} \,. \quad \text{(A.4)}$$

2. As representative of another class of autocorrelation functions, we take

$$B(\tau) = \sigma^2 e^{-c\tau} \cos \alpha\tau \,. \quad \text{(A.5)}$$

Now

$$\begin{cases} \tau_{\rm s} = \dfrac{c}{c^2+\alpha^2} \,, \\ \tilde{B}(\tau) = \dfrac{k\sigma^2 e^{-c\tau}}{\alpha^2+(c+k)^2}[(c+k)\cos\alpha\tau - \alpha\sin\alpha\tau] \,, \\ \tilde{\tau}_{\rm s} = \int_0^\infty \tilde{R}(\tau)d\tau = \tau_{\rm s}\left[1 - \dfrac{\alpha^2}{c(c+k)}\right] < \tau_{\rm s} \,. \end{cases} \quad \text{(A.6)}$$

The model expressions (A.1) and (A.5) are often used in meteorological applications.

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