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On Nonlinear Stability Theorems of 3D Quasi-geostrophic Flow

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ABSTRACT

Nonlinear stability criteria for quasi-geostrophic zonally symmetric flow are improved by establishing an optimal Poincaré inequality. The inequality is derived by a variational calculation considering the additional invariant of zonal momentum. When applied to the Eady model in a periodic channel with finite zonal length, the improved nonlinear stability criterion is identical to the linear normal-mode stability criterion provided the channel meridional width is no greater than $0.8605\cdots$ times its channel length (which is the geophysically relevant case).

Key words: 3D-Quasi-geostrophic flow, Nonlinear stability, Eady model

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1. Introduction

The energy-Casimir method has been used to obtain nonlinear stability of multilayer two-dimensional quasi-geostrophic flow (c.f., Liu and Mu 1992, 1994), and of three-dimensional quasi-geostrophic in spherical geometry (c.f., Li and Mu, 1996), and even of a two-layer shallow water semi-geostrophic model by Ren (2005), where the Poincaré inequalities play an important role.

Although McIntyre and Shepherd (1987), Mu and Wang (1992), and Mu and Simon (1993) established Arnold's second type of nonlinear stability for continuously stratified quasi-geostrophic motions under homogeneous and inhomogeneous boundary conditions, their results cannot be applied to the Eady model, since the basic state of the Eady model is not a stationary point of the functional constructed by the conservation of energy and zonal momentum. Mu and Shepherd (1994) overcame this difficulty by using conservation of the perturbed potential enstrophy in the Eady model and established a nonlinear stability criterion. Liu and Mu (1996) improved this result by a variational calculation. Then Liu et al. (1996) (referred to hereafter as LMS) recovered the result of Liu and Mu (1996) as a special application of a general nonlinear stability theorem for continuously stratified quasi-geostrophic flow, and also established Liapunov stability. When the zonal length is infinite, the nonlinear stability criterion is the same as the linear criterion. However the LMS criterion is independent of channel length and differs from the linear criterion for a finite channel.

Generally speaking, linear stability does not always imply a nonlinear one. And the extension of the linear result to a nonlinear one is difficult. For two-dimensional quasi-geostrophic flow in a channel, by considering the invariant of zonal momentum (Shepherd, 1989), the nonlinear stability criteria for multilayer flow are improved (c.f., Liu, 1999), and the nonlinear stability criterion for one layer flow is proved to be identical to the linear one provided the zonal length of the channel is no less than $2/\sqrt{3}(\approx 1.1547)$ times its meridional width (c.f., Liu and Li, 2003).

For three-dimensional flow, we consider a minimizational problem in three-dimensional space rather than a minimization problem in two lower dimensional spaces as in LMS. Considering the invariant of zonal momentum, the technique of LMS is developed and the Poincaré inequality of LMS (4.16) is improved to an optimal one for quasi-geostrophic parallel flow in a periodic channel with finite zonal length. Following LMS, two nonlinear stability criteria are established. When applied to the Eady model (c.f., LMS or Liu and

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Mu, 1996), the improved nonlinear stability criterion is exactly the linear normal-mode stability criterion provided the channel meridional width is no greater than $0.8605\cdots$ times its zonal length (which is the geophysically relevant case).

2. The model

We consider the standard three-dimensional, continuously stratified quasi-geostrophic flow on a beta-plane (e.g., Pedlosky, 1987). The governing equations are

$$\frac{DP}{Dt} \equiv P_t + \partial(\Phi, P) = 0 \,,$$
$$P \equiv \nabla^2\Phi + \frac{1}{\rho}(r\Phi_z)_z + f + \beta y \,, \tag{1}$$

where $P = P(t,x,y,z)$ is the potential vorticity, $\Phi = \Phi(t,x,y,z)$ is the stream function, $\partial(F,G) \equiv F_xG_y - F_yG_x$ is the horizontal Jacobian operator, x and y are zonal and meridional coordinates, respectively, t and is the time, and $\nabla = (\partial_x, \partial_y)$. Here $\rho = \rho(z) > 0$ is the prescribed reference-state density; f is the constant Coriolis parameter; β is the constant planetary vorticity gradient; $r = r(z) \equiv \rho/S$, where S is the prescribed reference-state static stability.

The boundary conditions on the lower and upper horizontal surfaces $z = z_1, z_2$ are

$$\frac{D\Lambda_i}{Dt} \equiv \Lambda_{it} + \partial(\Phi_i, \Lambda_i) = 0, \text{ on } z = z_i \ (i=1,2) \,, \tag{2}$$

where $\Lambda_i \equiv \Phi_{zi} + fS_i\eta_i$, and η_i is the topography, if there is any (normally $\eta_2 = 0$). Here the subscript $i = 1, 2$ denotes the value on $z = z_i$.

In the following we need only derive the improvement of Eq. (4.16) in LMS for the zonally symmetric case of the standard quasi-geostrophic model.

3. Improvements of the nonlinear stability criteria

Suppose that $(P, \Phi, \Lambda_1, \Lambda_2) = (Q, \Psi, \Theta_1, \Theta_2)$ is the basic steady state, and define the disturbance by $(q, \psi, \theta_1, \theta_2) = (P - Q, \Phi - \Psi, \Lambda_1 - \Theta_1, \Lambda_2 - \Theta_2)$. In the derivation of the nonlinear stability criteria [LMS (5.24), (6.11)] for continuously stratified quasi-geostrophic flow, the key Poincaréinequality [LMS (4.16)] is the following:

$$\mathcal{E}[\psi'] \geqslant [\lambda_0 + \mu_0(K)] \iiint_\Omega \frac{\rho}{2}(\psi')^2 dxdydz + \sum_{i=1}^{2} \frac{K}{C_i} \iint_D \frac{\rho_i}{2S_i}(\psi_i')^2 dxdy \,, \tag{3}$$

where $\psi' \equiv \psi - \psi(0,x,y,z)$, $\mathcal{E}[\psi']$, is defined by

$$\mathcal{E}[\psi'] = \iiint_\Omega \frac{\rho}{2}\left\{|\nabla\psi'|^2 + \frac{1}{S}\psi_z'^2\right\}(\psi')^2 dxdydz;$$

the domain $\Omega \equiv D \times [z_1, z_2]$, the horizontal domain D is bounded by J smooth simple closed curves ∂D_j; $j = 1, \ldots, J$; the positive constants C_1 and C_2 are defined by LMS, (3.2b); the subscript $i = 1, 2$ denotes the value on $z = z_i$; $\nabla \equiv (\partial_x, \partial_y)$; K is an arbitrary positive constant. We see that ψ' satisfies the following conditions [LMS (4.4), (4.5)]:

$$\psi_s' = 0 \text{ and } \oint_{\partial D_j} \psi_n' ds = 0 \text{ on } \partial D_j \ (j = 1, \ldots, J) \,, \tag{4}$$

$$\iint_D \psi' dxdy = 0 \quad \forall z \in [z_1, z_2] \,, \tag{5}$$

where the subscripts s and n refer respectively to the tangential and normal derivatives on the curves ∂D_j. Finally, λ_0 is the lowest non-trivial eigenvalue of the problem.

$$\nabla^2 u + \lambda_0 u = 0 \quad \text{in} \quad D \tag{6}$$

with the same boundary conditions (4) as applied to ψ'; and $\mu_0(K)$ is the lowest eigenvalue of the Sturm-Liouville eigenvalue problem [cf. LMS (4.13), (4.14), where is now denoted by λ_0]

$$\left(\frac{\rho}{S}v_z\right)_z + \mu(K)\rho(z)v = 0 \,,$$
$$C_i v_z(z_i) = (-1)^i K v(z_i) \ \ (i = 1, 2) \,. \tag{7}$$

In fact, $\lambda_0 + \mu_0(K)$ is exactly the minimum of the minimization problem

$$\lambda(K) \equiv \min \frac{\mathcal{E}[\psi'] - \sum_{i-1}^{2} \frac{K}{C_i} \iint_D \frac{\rho_i}{2S_i}(\psi_i')^2 dxdy}{\iiint_\Omega \frac{\rho}{2}(\psi')^2 dxdydz}$$
$$\equiv \min \frac{\mathcal{E}[\psi']}{\mathcal{E}[\psi']} \,, \tag{8}$$

subject to the constraints (4)–(5).

When the problem is not zonally symmetric, the inequality (3) is optimal since the equality holds when we take $\psi' = u_1(x,y)v_0(z)$, where $u_1(x,y)$ is the second eigenfunction of (6) (the first eigenfunction is a non-zero constant corresponding to the trivial eigenvalue zero) and $v_0(z)$ the first eigenfunction of (7). Therefore, the result of LMS cannot be improved in nonzonal geometry by the energy-Casimir method.

But in the zonally symmetric case, $\lambda(K)$ is larger since in (7) ψ' has an additional constraint derived by the conservation of zonal impulse [c.f. LMS (2.8)]:

$$M[\Phi] = \iiint_\Omega \rho y P dxdydz - \iint_D r_i y \Lambda_i dxdy \Big|_{i=1}^{i=2} .$$

That is

$$M[\Phi] - M[\Phi_0] = \iiint_\Omega \rho y q' dxdydz - \iint_D r_i y \psi'_z dxdy_i \Big|_{i=1}^{i=2} = 0 .$$

Note that

$$q' = \nabla^2 \psi' + \frac{1}{\rho}(r\psi'_z)_z ,$$

then integration by parts gives

$$M[\Phi] - M[\Phi_0] = \iiint_\Omega \rho y \nabla^2 \psi dxdydz = 0 .$$

Using Green's formula with condition (4), we have the additional constraint of ψ':

$$\iiint_\Omega \rho \psi'_y dxdydz = 0 . \tag{9}$$

So, now we redefine $\lambda(K)$ by (7) with constraints (4), (5) and (9). Notice that this newly defined $\lambda(K)$ is also a continuous decreasing function of K for $K \geqslant 1$.

If $\lambda(1) > 1/C_3$, then there exists a unique $K > 1$ such that $\lambda(K) = a > 0$ which is similar to LMS (4.17), and the nonlinear stability follows similar to LMS. If $\lambda(1) > 0$, then by the continuity of $\lambda(K)$, there exist constants $K > 1$ and $a > 0$ such that $\lambda(K) = a > 0$, which is similar to LMS (6.3), and the nonlinear stability follows similar to LMS. Thus we can write the two new distinct nonlinear stability criteria in the forms

$$\lambda(1) > \frac{1}{C_3} \text{ and } \lambda(1) > 0 , \tag{10}$$

respectively, where $\lambda(1)$ is now calculated by taking $K = 1$ in (8) with the constraints (4), (5) and (9); the constant C_3 is defined by LMS (3.2a). The latter criterion can be applied to a flow with horizontally uniform potential vorticity, and the criterion is better because $\lambda(1)$ is larger when more constraints are imposed on the minimization problem (8).

4. Application to the Eady model

Using the approach of LMS, Liu and Mu (2001) established both linear and nonlinear stability theorems for the generalized Eady model in a finite periodic channel, and found that the nonlinear and linear stability criteria differ by a term involving the channel length. By using the latter nonlinear stability criterion of (10), the result of Liu and Mu (2001) can be improved. But for simplicity, we discuss the Eady model only. The discusson of any particular generalized Eady model is similar.

The Eady problem of baroclinic instability is a classical one in geophysical fluid dynamics (e.g. Pedlosky, 1987, §7.7). For the Eady model: $\rho = 1$, $\beta = 0$, $S > 0$ is constant. The Eady basic state is $\Psi = -syz$, $Q = f$, $\Theta_i = -sy$, which represents a basic flow with vertical shear s. We consider the horizontal domain $D = \{(x, y)|x \in [-X, X],\ y \in [-Y, Y]\}$, namely a periodic channel with finite zonal length $2X$, width $2Y$ and height $2H = z_2 - z_1$; without loss of generality, we may let $z_1 = -H$, $z_2 = H$. Then $C_1 = C_2 = H$, $\mu_0(1) = -w_0^2/(H^2 S)$, where $w_0 \approx 1.19967864$ is the positive root of $w \tanh w = 1$. The linear stability criterion for the generalized Eady model has been derived by Liu and Mu (2001), and the linear stability criterion for the Eady model is a special case:

$$\frac{\pi^2}{4Y^2} + \frac{\pi^2}{X^2} - \frac{w_0^2}{H^2 S} > 0 . \tag{11}$$

The linear stability criterion (11) is derived by linearizing the governing equations, and taking solutions in the normal mode form:

$$\begin{aligned}\psi =& v(z)\cos[(j + 1/2)\pi/Y] \times \\ & \exp(ni\pi[x - s(c - H)t]/X) , \\ & j = 0, 1, 2 \ldots ; n = 1, 2, \ldots\end{aligned}$$

(for the details, see Liu and Mu, 2001). The nonlinear stability criterion in LMS is

$$\lambda_0 + \mu_0(1) = \frac{\pi^2}{4Y^2} - \frac{w_0^2}{H^2 S} > 0 .$$

And our new nonlinear stability criterion (to be proved in Appendix A) is:

$$\min\left(\frac{\pi^2}{4Y^2} + \frac{\pi^2}{X^2}, \frac{0.990\ldots\pi^2}{Y^2}\right) - \frac{w_0^2}{H^2 S} > 0 . \tag{12}$$

Thus, for a finite channel, our criterion is stronger than that of LMS; and if $Y/X \leqslant 0.8605\ldots$ (which is the geophysically relevant case), then the nonlinear stability criterion (10) is the same as the linear stability criterion (11).

5. Discussion

Based on the work of Liu et al. (1996), we reexamined the nonlinear stability for standard continously stratified quasi-geostrophic flow, and we presented the nonlinear stability criteria (10) in the inequalities of eigenvalues for certain problems. And for the zonally symmetric flow, we improved the result of Liu et al. (1996) by imposing zonal momentum invariance in the variational calculation. The new result is better because the eigenvalue in (10) is larger if more constraints are imposed. Though we only take the Eady model as an example of application, the methods used in the Appendixes A and B can be also applied to other models, such as the generalied Eady model (c.f. Liu and Mu, 2001).

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APPENDIX A

Proof of (12)

Let

$$\eta(y,z)=\frac{1}{2X}\int_{-X}^{X}\psi' dx,\ \ \xi(x,y,z)=\psi'-\eta(y,z)\ . \tag{A1}$$

Then by (A1), we have

$$\mathcal{G}[\psi']=\mathcal{G}[\xi]+\mathcal{G}[\eta],\ \ \mathcal{F}[\psi']=\mathcal{F}[\xi]+\mathcal{F}[\eta]\ , \tag{A2}$$

where $\mathcal{G}$ and $\mathcal{F}$ are defined by (8). By (4)–(5), (A1) and (9), we have the following constraints on ξ and η:

$$\int_{-X}^{X}\xi dx=0,\ \xi \text{ is periodic in } x \text{ with period } 2X\ ,$$

$$\xi(x,\pm Y,z)=0\ , \tag{A3}$$

$$\eta_y(\pm Y,z)=0,\ \ \int_{-Y}^{Y}\eta dy=0\ , \tag{A4}$$

$$\int_{z_1}^{z_2}\frac{\rho(z)}{2}[\eta(Y,z)-\eta(-Y,z)]dz=0\ , \tag{A5}$$

where (A5) is the constraint derived from (9).

For the Eady model, within a constant normalization factor, the eigenfunction $\varphi_j=\varphi_j(z)$ of (7) with $K=1$ corresponding to eigenvalue $\mu_0(1)=-w_0^2/(H^2S)$, $\mu_j(1)=w_j^2/(H^2S)$, $j>0$ is:

$$\varphi_0=\frac{\cosh\left(\frac{w_0z}{H}\right)}{\cosh(w_0)},\ \ w_0\tanh w_0=1\ , \tag{A6}$$

$$\varphi_1=\sqrt{\frac{3}{2}}\frac{z}{H},\ \ \mu_1(1)=0\ . \tag{A7}$$

$$\varphi_{2j}=\frac{\cos\left(\frac{w_{2j}z}{H}\right)}{\cos(w_{2j})}\ ,$$

$$-\cot w_{2j}=w_{2j}\in\left[\left(j-\frac{1}{2}\right)\pi, j\pi\right]\ , \tag{A8}$$

$$\varphi_{2j}=\frac{\sin\left(\frac{w_{2j+1}z}{H}\right)}{\sin(w_{2j+1})}\ ,$$

$$\tan w_{2j+1}=w_{2j+1}\in\left[j\pi,\left(j+\frac{1}{2}\right)\pi\right]\ . \tag{A9}$$

Then by (A3), $\xi(x,y,z)$ can be expanded by a series of orthogonal functions:

$$\exp\left(\frac{im\pi x}{X}\right)\sin\left(\frac{n(y-Y)\pi}{2Y}\right)\varphi_k(z)\ ,$$

$$m,n=1,2,\ldots;\ k=0,1,2,\ldots\ .$$

Therefore we have an optimal inequality:

$$\mathcal{F}[\xi]\geqslant\left(\frac{\pi^2}{4Y^2}+\frac{\pi^2}{X^2}-\frac{w_0^2}{H^2S}\right)\mathcal{G}[\xi]\ . \tag{A10}$$

In the same way, by (A4), we have the orthonormal-function expansion of η:

$$\eta(y,z)=\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}c_{k,n}\cos\left(\frac{n(y-Y)\pi}{2Y}\right)\varphi_k(z)\ , \tag{A11}$$

and the constraint (A5) on η can be written as:

$$\sum_{k=0}^{\infty}a_k\sum_{n=1}^{\infty}c_{k,2n-1}=0\ \ a_k\equiv\frac{1}{z_2-z_1}\int_{z_1}^{z_2}\rho\varphi_k(z)dz\ . \tag{A12}$$

We can see that, for the Eady model, $a_{2j+1}=0$ for all $j\geqslant 0$ by (A7) and (A9); $a_0=1/w_0^2$ and $a_{2j}=-1/w_{2j}^2$ for $j\geqslant 1$ by (A6) and (A8).

Now by (A11), we change the minimization problem of η to a discrete one:

$$\mu\equiv\min\frac{\mathcal{F}[\eta]}{\mathcal{G}[\eta]}=\min\frac{\sum\limits_{k=0}^{\infty}\sum\limits_{n=1}^{\infty}\lambda_{kn}c_{k,n}^2}{\sum\limits_{k=0}^{\infty}\sum\limits_{n=0}^{\infty}c_{k,n}^2}\ , \tag{A13}$$

subject to constraint (A12), where

$$\lambda_{k,n}=\frac{n^2\pi^2}{4Y^2}+\mu_k(1)\quad k=0,1,\ldots;\ \ n=1,2,\ldots\ . \tag{A14}$$

Combining (A2), (A10) and (A13), we have an optimal inequality for the Eady model:

$$\mathcal{F}[\psi']\geqslant\min\left(\frac{\pi^2}{4Y^2}+\frac{\pi^2}{X^2}-\frac{w_0^2}{H^2S},\mu\right)\mathcal{G}[\psi']$$
$$=\lambda(1)\mathcal{G}[\psi']\ ,$$

where $\lambda(1)$ is equal to the value of (8) subject to the constraints (4)–(5) and (9) with $K=1$ and $C_1=C_2=H$. Thus, the nonlinear stability criterion is

$$\lambda(1)=\min\left(\frac{\pi^2}{4Y^2}+\frac{\pi^2}{X^2}-\frac{w_0^2}{H^2S},\mu\right)>0\ . \tag{A15}$$

We shall prove (in Appendix B) that:

$$\text{If }\ \frac{0.990589\cdots\pi^2}{Y^2}-\frac{w_0^2}{H^2S}>0\ ,\text{ten }\ \mu>0\ . \tag{A16}$$

Thus, the criterion (12) follows from (A15) and (A16).

APPENDIX B

Proof of (A16)

We discuss a discrete conditional minimization problem:

Let λ_j and s_j be two given real sequences such that $\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \lambda_j$, for all $j > 3$.

Denoting

$$\mu(\boldsymbol{c}) \equiv \frac{\sum\limits_{j=1}^{\infty} \lambda_j c_j^2}{\sum\limits_{j=1}^{\infty} c_j^2}\,, \tag{B1}$$

where $\boldsymbol{c} = \{c_j\} \neq 0$ subjected to the condition

$$\sum_{j=1}^{\infty} c_j s_j = 0\,, \tag{B2}$$

then we have the following lemma.

Lemma 1. When $s_1 \neq 0$, and if there is a $\mu \in (\lambda_1, \lambda_2)$ satisfying

$$\sum_{j=1}^{\infty} \frac{s_j^2}{\lambda_j - \mu} = 0\,, \tag{B3}$$

then $\min\limits_{\boldsymbol{c}} \mu(\boldsymbol{c}) = \mu$.

Proof. Taking

$$c_j = \frac{s_j^2}{\lambda_j - \mu},\ \ j = 1, 2, \ldots$$

in (B1), we can see that $\mu(\boldsymbol{c}) = \mu$ by (B3). On the other hand, if $c_1 = 0$, we see that $\mu(\boldsymbol{c}) \geqslant \lambda_2$ by (B1). When $c_1 \neq 0$, by constraint (B2), the Chauchy inequality and (B3),

$$\begin{aligned}
1 &= -\sum_{j=2}^{\infty} \frac{s_j c_j}{s_1 c_1} = -\sum_{j=2}^{\infty} \frac{s_j\sqrt{\mu-\lambda_1}}{s_1\sqrt{\lambda_j-\mu}} \frac{c_j\sqrt{\lambda_j-\mu}}{c_1\sqrt{\mu-\lambda_1}} \\
&\leqslant \sqrt{\sum_{j=2}^{\infty} \frac{s_j^2(\mu-\lambda_1)}{s_1^2(\lambda_j-\mu)}} \sqrt{\sum_{j=2}^{\infty} \frac{c_j^2(\lambda_j-\mu)}{c_1^2(\mu-\lambda_1)}} \\
&= \sqrt{\sum_{j=2}^{\infty} \frac{c_j^2(\lambda_j-\mu)}{c_1^2(\mu-\lambda_1)}}
\end{aligned}$$

That is,

$$1 \leqslant \sqrt{\sum_{j=2}^{\infty} \frac{c_j^2(\lambda_j-\mu)}{c_1^2(\mu-\lambda_1)}}$$

which implies $\mu(c) \geqslant \mu$. Therefore, $\min\limits_{\boldsymbol{c}} \mu(\boldsymbol{c}) = \mu$.

Now we apply Lemma 1 to problem (A13) subject to (A12) with $\lambda_1 = \lambda_{01} = \pi^2/(4Y^2) + \mu_0(1)$, $c_1 = c_{0,1}$, $s_1 = a_0 \neq 0$, $\lambda_2 = \min(\lambda_{02},\ \lambda_{11})$, $\lambda_3 = \max(\lambda_{02},\ \lambda_{11})$, and arrange the other λ_{kn} of (A14) in any order by $\lambda_3, \lambda_4, \ldots$.

In the interval $I \equiv (\lambda_1, \lambda_2)$ we consider Eq. (B3), which is now to be

$$\sum_{k=0}^{\infty} \frac{1}{\mu_{2k}^2(1)} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2 \dfrac{\pi^2}{4Y^2} + \mu_{2k}(1) - \mu} = 0\,. \tag{B4}$$

By the theory of complex analysis, we have the identity (which holds for any complex number z)

$$\frac{\tan z}{2z} \equiv \frac{\tanh(iz)}{2iz} \equiv \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2\pi^2/4 - z^2}\,.$$

Eq. (B4) can be written as

$$F(\mu, Y) \equiv \frac{\tan\left(\sqrt{\mu + \dfrac{w_0^2}{H^2 S}}Y\right)}{\mu_0^2(1)\sqrt{\mu + \dfrac{w_0^2}{H^2 S}}} + \sum_{k=1}^{\infty} \frac{\tanh(\sqrt{\mu_{2k}(1) - \mu}Y)}{\mu_{2k}^2(1)\sqrt{\mu_{2k}(1)-\mu}} = 0\,, \tag{B5}$$

where $\tanh(\sqrt{x})/\sqrt{x} = \tan(\sqrt{-x})/\sqrt{-x}$ if $x < 0$, and we define $\tanh(0)/0 = 1$. We see that the function $F(\mu, Y)$ defined by (B5) is a continuous increasing function both of $Y > 0$ and $\mu \in I$.

In the following, we need only consider the case that $\lambda_1 \leqslant 0$ (otherwise $\mu \geqslant \lambda_1 > 0$ by (B1). It is easy to verify that $\lim\limits_{\mu\to\lambda_1+0} F(\mu, Y) < 0$ and $\lim\limits_{\mu\to\lambda_2-0} F(\mu, Y) > 0$. Therefore, (B5) has a unique root μ in the interval I by the continuity and monotonicity of $F(\mu, Y)$ in μ.

By the monotonicity of $F(0, Y)$ in Y, we found by numerical computation that if

$$\frac{0.990589\cdots\pi^2}{Y^2} - \frac{w_0^2}{H^2 S} > 0\,,$$

then $\lambda_2 > 0$, and $F(0, Y) < 0$. Hence there is a positive root μ of (B5) in the interval $(0, \lambda_2) \subset (\lambda_1, \lambda_2)$. This completes the proof of (A16).

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