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On the Generalized Ertel–Rossby Invariant

GAO Shouting*[1,4] (高守亭), XU Pengcheng[2] (许鹏程), RAN Lingkun[1] (冉令坤), and LI Na[1,3] (李 娜)

[1] *Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing* 100029

[2] *Institute of Applied Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing* 100190

[3] *Graduate University of Chinese Academy of Sciences, Beijing* 100040

[4] *State Key Laboratory of Severe Weather, Chinese Academy of Meteorological Sciences, Beijing* 100081

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ABSTRACT

A new invariant called the generalized Ertel–Rossby invariant (GER) was developed in this study. The new invariant is given by the dot product of the generalized vorticity and the generalized velocity. The generalized vorticity is the absolute vorticity minus the cross–product of the gradient of Lagrangian–time integrated temperature and the gradient of entropy. The generalized velocity is the absolute velocity minus the sum of the gradient of Lagrangian–time integrated kinetic potential and the Lagrangian–time integrated temperature multiplied by the gradient of entropy. In addition to the traditional potential vorticity, the GER invariant may provide another useful tool to study the atmospheric dynamic processes for weather phenomena ranging from large scales to small scales.

Key words: potential vorticity, generalized Ertel–Rossby invariant, generalized vorticity, generalized velocity

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1. Introduction

Potential vorticity is an important invariant and a conserved tracer for an adiabatic and frictionless flow. It was initially given by Rossby (1936) for a barotropic atmosphere, and its exact form was given by Ertel (1942) for a baroclinic atmosphere, which is

$$\mathrm{PV} = \rho^{-1}\boldsymbol{\xi}_{\mathrm{a}} \cdot \nabla\theta \ ,$$

where ρ denotes the mass density per unit volume, $\boldsymbol{\xi}_{\mathrm{a}} = \boldsymbol{\xi} + 2\boldsymbol{\Omega}$ is the absolute vorticity, $\boldsymbol{\xi} = \nabla \times \boldsymbol{V}$ is the relative vorticity, $\boldsymbol{\Omega}$ is the angular velocity of the earth, $\boldsymbol{V}$ is the air velocity relative to the Earth, ∇ is the three–dimensional gradient operator to $\boldsymbol{r}$, and θ is the potential temperature.

The most important property of PV is its material conservation for an adiabatic and frictionless flow, and this property is very useful for visualizing the atmospheric motion and studying the related dynamic processes. Thus PV has been applied for several purposes: (1) to characterize the two–dimensional motion in a layered manner; (2) to take account of phenomena such as the often–temporary spin–up or spin–down caused by adiabatic vertical motion; (3) to describe the important aspects of the dynamics without explicit reference to the vertical motion; and (4) to visualize and comprehend quasi–horizontal, two–dimensional advection (Uccellini et al., 1985; Hoskins and Berrisford, 1988; Hoskins, 1997; Holton, 2004).

A related property of the potential vorticity for large–scale motions is its invertibility under a suitable balance condition, which assumes that gravity and inertial–gravity waves are either absent altogether or can be averaged out. Sometimes it is referred to a "slow–manifold" condition (Egger, 1990; McIntyre and Norton, 1991). Therefore, with its conservation and invertibility, the approximated PV and related "PV–

*Corresponding author: GAO Shouting, gst@lasg.iap.ac.cn

thinking" (Hoskins et al., 1985) have been particularly useful for simplified understanding and interpretation of various large–scale atmospheric processes in which the layered two–dimensional and non–gravity wave parts of the motion are primary. However, the physical variables (e.g., velocity, pressure) would change rapidly for severe weather such as cyclones, vortices, or thunderstorms. Thus the adoption of the balance approximation becomes questionable. As we know, potential vorticity is the dot product of absolute vorticity and the gradient of potential temperature, so it cannot provide a complete description of the motion whose velocity and dynamic pressure vary rapidly and locally. Hence, it is quite necessary to find another materially conserved invariant, which may include additional dynamical information than the potential vorticity for the baroclinic atmosphere.

By using the Weber transformation and the Lagrangian continuity equation, Ertel and Rossby (1949, ER49 hereafter) obtained an invariant different from PV, which is called the Ertel–Rossby (ER) invariant in the barotropic atmosphere. Zdunkowski and Bott (2003, ER03 hereafter) also used the Weber transformation and obtained the Ertel–Rossby invariant in the non–rotating baroclinic atmosphere, called the baroclinic Ertel–Rossby invariant. However, the invariant in the rotating framework was only briefly mentioned in ER03, with no detailed derivation. Moreover, the Webber transformation used in the derivation of ER49 and ER03 was only a mathematical transformation, which did not include much physical background.

The generalized vorticity equation (Mobbs, 1981; Wu, 2002) provided another way to obtain some invariants for atmospheric dynamics. By defining a tracer function, a new invariant could be obtained. For example, if the tracer function was the potential temperature, the traditional potential vorticity could be deduced. Hence, to find a new invariant to describe the motion of the atmosphere, another tracer function, different from the potential temperature, needs to be defined.

For this purpose, the Clebsch transformation was employed to define a new tracer function to obtain a new invariant. With the Clebsch transformation the motion equation was transformed into a velocity field equation. Based on the velocity field equation, we defined a new tracer function and deduced another invariant, which was named the generalized Ertel–Rossby (GER) invariant. The new invariant may not only describe dynamics on the slow manifold but may also include dynamical information of severe weather.

This paper is arranged as follows. In Section 2, we introduce the generalized vorticity equation. In Section 3, the Clebsch transformation is used to find the new invariant GER for the baroclinic atmosphere in the rotating frame. A concluding remark is given in Section 4.

2. Generalized vorticity equation

In this study, we considered the motion of a compressible, adiabatic and non–dissipative atmosphere in a region D. The motion equation for an air particle is given by

$$d_t\boldsymbol{V} + 2\boldsymbol{\Omega}\times\boldsymbol{V} + 1/\rho\nabla p + g\boldsymbol{k} = 0\,, \qquad (1)$$

where $\boldsymbol{V}$ is the relative velocity, $\boldsymbol{\Omega}$ is the angular velocity of the rotating frame, g is the gravity, p is the pressure, ρ is the density, $\boldsymbol{k}$ is the unit vector along the earth radius in the local Cartesian coordinates.

The entropy of the air particle S is defined as $S = c_p\ln\theta + C$ (c_p is the specific heat at constant pressure and C is a constant). So

$$\nabla S = \frac{c_p}{\theta}\nabla\theta = \frac{c_p}{T}\nabla T - \frac{R}{p}\nabla p\,. \qquad (2)$$

Hence the pressure gradient force in Eq. (1) can be rewritten as

$$-1/\rho\nabla p = T\nabla S - \nabla c_pT = T\nabla S - \nabla H\,, \qquad (3)$$

where $H = c_pT = c_vT + p/\rho$ is the enthalpy where c_v is the specific heat at constant volume.

Substituting Eq. (3) into Eq. (1) yields another form of the motion equation

$$d_t\boldsymbol{V} + 2\boldsymbol{\Omega}\times\boldsymbol{V} - T\nabla S + \nabla H + \nabla\phi = 0, \qquad (4)$$

where $\nabla\phi = g\boldsymbol{k}$ and ϕ is the geopotential. Then the vorticity equation is given by the curl of Eq. (4) as

$$d_t(\boldsymbol{\xi}_{\rm a}/\rho) = (\boldsymbol{\xi}_a/\rho)\cdot\nabla\boldsymbol{V} - \nabla T\times\nabla S/\rho\,, \qquad (5)$$

For the barotropic atmosphere, Eq. (5) is

$$d_t(\boldsymbol{\xi}_{\rm a}/\rho) = (\boldsymbol{\xi}_{\rm a}/\rho)\cdot\nabla\boldsymbol{V}\,.$$

For the baroclinic atmosphere, the similar vorticity equation can be obtained as

$$d_t(\boldsymbol{\xi}_{\rm g}/\rho) = (\boldsymbol{\xi}_{\rm g}/\rho)\cdot\nabla\boldsymbol{V}\,, \qquad (6)$$

which is called the generalized vorticity equation. $\boldsymbol{\xi}_{\rm g} = \boldsymbol{\xi}_{\rm a} - \nabla\eta\times\nabla S$ is the generalized vorticity. η is the Lagrangian–time integrated temperature (T) along the trajectory of the air particle which can be expressed as $d_t\eta = T$.

The left hand part of Eq. (6) can be rewritten as

$$d_t\left(\frac{\boldsymbol{\xi} + 2\boldsymbol{\Omega} - \nabla\eta\times\nabla S}{\rho}\right) = d_t\left(\frac{\boldsymbol{\xi} + 2\boldsymbol{\Omega}}{\rho}\right) - d_t\left(\frac{\nabla\eta\times\nabla S}{\rho}\right)\,.$$

Expanding term $d_t\left(\frac{\nabla\eta\times\nabla S}{\rho}\right)$ obtains

$$d_t\left(\frac{\nabla\eta\times\nabla S}{\rho}\right)=-\frac{1}{\rho^2}\nabla\eta\times\nabla S\frac{d\rho}{dt}+\frac{1}{\rho}\frac{d}{dt}\nabla\eta \times\nabla S+\frac{1}{\rho}\nabla\eta\times\frac{d}{dt}\nabla S. \tag{7}$$

Using equations

$$\begin{aligned}\frac{d\rho}{dt}&=-\rho\nabla\cdot\boldsymbol{V}\ ,\\ \frac{d\nabla S}{dt}&=\nabla\frac{dS}{dt}-\nabla S\cdot\nabla\boldsymbol{V}-\nabla S\times\nabla\times\boldsymbol{V}\ ,\\ \frac{d\nabla\eta}{dt}&=\nabla\frac{d\eta}{dt}-\nabla\eta\cdot\nabla\boldsymbol{V}-\nabla\eta\times\nabla\times\boldsymbol{V}\ ,\end{aligned}\tag{8}$$

together with $d_tS=0$, $d_t\eta=T$ and the following identity:

$$\begin{aligned}&\nabla S\times(\nabla\eta\cdot\nabla)\boldsymbol{V}-\nabla\eta\times(\nabla S\cdot\nabla)\boldsymbol{V}+\\ &\nabla S\times[\nabla\eta\times(\nabla\times\boldsymbol{V})]-\nabla\eta\times[\nabla S\times(\nabla\times\boldsymbol{V})]\\ &=(\nabla S\times\nabla\eta)\nabla\cdot\boldsymbol{V}-(\nabla S\times\nabla\eta)\cdot\nabla\boldsymbol{V}\ .\end{aligned}\tag{9}$$

Eq. (7) can be rewritten as

$$d_t\left(\frac{\nabla\eta\times\nabla S}{\rho}\right)=\left(\frac{\nabla\eta\times\nabla S}{\rho}\right)\cdot\nabla V+\frac{\nabla T\times\nabla S}{\rho}. \tag{10}$$

Eq. (5) minus Eq. (10) yields Eq. (6).

From Eq. (6), we can prove that the operators d_t and $(\boldsymbol{\xi}_{\rm g}/\rho)\cdot\nabla$ commute for any tracer function λ, that is, the following equation would satisfy:

$$d_t\left[(\boldsymbol{\xi}_{\rm g}/\rho)\cdot\nabla\lambda\right]=(\boldsymbol{\xi}_{\rm g}/\rho)\cdot\nabla d_t\lambda \tag{11}$$

In particular, it is easy to see that

$$\begin{aligned}&d_t\left[(\boldsymbol{\xi}_{\rm g}/\rho)\cdot\nabla\lambda\right]\\ &=d_t\left[(\boldsymbol{\xi}_{\rm g}/\rho)\right]\cdot\nabla\lambda+(\boldsymbol{\xi}_{\rm g}/\rho)\cdot d_t\nabla\lambda\\ &=\left[(\boldsymbol{\xi}_{\rm g}/\rho)\cdot\nabla\boldsymbol{V}\right]\cdot\nabla\lambda+(\boldsymbol{\xi}_{\rm g}/\rho)\cdot d_t\nabla\lambda\\ &=\left[(\boldsymbol{\xi}_{\rm g}/\rho)\cdot\nabla\boldsymbol{V}\right]\cdot\nabla\lambda+(\boldsymbol{\xi}_{\rm g}/\rho)\cdot\\ &\quad(\nabla d_t\lambda-\nabla\lambda\cdot\nabla\boldsymbol{V}-\nabla\lambda\times\nabla\times\boldsymbol{V})\\ &=(\boldsymbol{\xi}_{\rm g}/\rho)\cdot\nabla d_t\lambda+(\boldsymbol{\xi}_{\rm g}/\rho)\cdot\\ &\quad(\nabla\lambda\cdot\nabla\boldsymbol{V}-\nabla\lambda\cdot\nabla\boldsymbol{V}-\nabla\lambda\times\nabla\times\boldsymbol{V})\\ &=(\boldsymbol{\xi}_g/\rho)\cdot d_t\nabla\lambda\ ,\end{aligned}\tag{12}$$

where Eq. (6) is used in the second step,

$$\begin{aligned}d_t\nabla\lambda&=\nabla d_t\lambda-\nabla(\boldsymbol{V}\cdot\nabla\lambda)+\boldsymbol{V}\cdot\nabla(\nabla\lambda)\\ &=\nabla d_t\lambda-\nabla\lambda\cdot\nabla\boldsymbol{V}-\nabla\lambda\times\nabla\times\boldsymbol{V}\end{aligned}$$

is used in the third step, and the identity

$$\nabla\boldsymbol{V}\cdot\nabla\lambda-\nabla\lambda\cdot\nabla\boldsymbol{V}=\nabla\lambda\times\nabla\times\boldsymbol{V}$$

is used in the last step.

The original derivation and related discussion of Eq. (11) can be found in Mobbs (1981) and Wu (2002).

3. Clebsch transformation and the generalized Ertel–Rossby invariant

Equation (11) provides a method of finding new invariants. For example, let the tracer function be the entropy of the atmospheric particle (S), from Eq. (11) one can obtain

$$\begin{aligned}&d_t\left[\left(\frac{\boldsymbol{\xi}_{\rm a}-\nabla\eta\times\nabla S}{\rho}\right)\cdot\nabla S\right]\\ &=d_t(\boldsymbol{\xi}_{\rm a}\cdot\nabla S/\rho)=\left(\frac{\boldsymbol{\xi}_{\rm a}-\nabla\eta\times\nabla S}{\rho}\right)\cdot\nabla d_tS.\end{aligned}\tag{13}$$

For an adiabatic flow, $d_tS=0$, so Eq. (13) can be written as

$$d_t(\boldsymbol{\xi}_{\rm a}\cdot\nabla S/\rho)=0. \tag{14}$$

From Eq. (14), the well–known potential vorticity

$$\mathrm{PV}=\boldsymbol{\xi}_{\rm a}\cdot\nabla S/\rho\ , \tag{15}$$

is then obtained. To describe the dynamics of severe weather, another tracer function, which would include much dynamical information, should be found. In this section, the Clebsch transformation is applied to find a new tracer function. By this transformation, the motion Eq. (4) was transformed as a velocity field equation, which includes nearly all dynamical information for the air flow.

The Clebsh transformation for the velocity

$$\boldsymbol{V}=\nabla\Phi+\alpha\nabla\beta$$

was first introduced by (Clebsch, 1859) and applied to barotropic flow only. Here, Φ, α and β are the integral of Lagrangian function. Physically, α and β can be the initial velocity field and Lagrangian displacement field, respectively, according to Dutton (1976). Herivel (1955) proposed a similar formula applying to non–barotropic flow:

$$\boldsymbol{V}=\nabla\Phi+\eta\nabla S\ ,$$

which is not of general validity. Seliger and Whitham (1968) indicated that for the general flow the velocity can be written as follows

$$\boldsymbol{V}=\nabla\Phi+\eta\nabla S+\alpha\nabla\beta, \tag{16}$$

where

$$\begin{cases}d_t\Phi=\frac{1}{2}\boldsymbol{V}^2-H-\phi\ ,\\ d_t\alpha=d_t\beta=0\ ,\\ d_t\eta=T\ ,\end{cases}\tag{17}$$

H is the enthalpy and ϕ is the geopotential.

For a rotating frame, we considered the Clebsch transformation for the absolute velocity defined by $\boldsymbol{V}_{\rm a}=\boldsymbol{V}+\boldsymbol{V}_{\rm e}=\boldsymbol{V}+\boldsymbol{\Omega}\times\boldsymbol{r}$ as follows:

$$\boldsymbol{V}_{\rm a}=\nabla\Phi+\eta\nabla S+\alpha\nabla\beta, \tag{18}$$

where $\boldsymbol{V}_{\rm a}$ is the absolute velocity, $\boldsymbol{V}_{\rm e}$ is the velocity due to the rotation of the earth and $d_t\alpha=d_t\beta=0$.

Differentiating Eq. (18) with respect to the time t yields

$$\begin{aligned}d_t\boldsymbol{V}_{\rm a}&=d_t\nabla\Phi+T\nabla S+\eta d_t\nabla S+\alpha d_t\nabla\beta\\&=\nabla d_t\Phi+T\nabla S-\nabla\Phi\cdot\nabla\boldsymbol{V}-\nabla\Phi\times(\nabla\times\boldsymbol{V})+\\&\quad\eta\nabla d_tS-\eta\nabla S\cdot\nabla\boldsymbol{V}-\eta\nabla S\times(\nabla\times\boldsymbol{V})+\\&\quad\alpha\nabla d_t\beta-\alpha\nabla\beta\cdot\nabla\boldsymbol{V}-\alpha\nabla\beta\times(\nabla\times\boldsymbol{V})\,.\end{aligned} \tag{19}$$

Substituting $\alpha\nabla\beta$ in (18) into (19) gives

$$\begin{aligned}d_t\boldsymbol{V}_{\rm a}&=\nabla d_t\Phi+T\nabla S-\boldsymbol{V}_{\rm a}\cdot\nabla\boldsymbol{V}-\boldsymbol{V}_{\rm a}\times(\nabla\times\boldsymbol{V})\\&=\nabla d_t\Phi+T\nabla S-\frac{1}{2}(\boldsymbol{V}_{\rm a}^2-\boldsymbol{V}_{\rm e}^2)+\boldsymbol{V}\times\boldsymbol{\Omega}\,.\end{aligned} \tag{20}$$

By noting $d_t\boldsymbol{V}_{\rm a}=d_t\boldsymbol{V}+d_t(\boldsymbol{\Omega}\times\boldsymbol{r})=d_t\boldsymbol{V}+\boldsymbol{\Omega}\times\boldsymbol{V}$ and comparing Eq. (20) with Eq. (4), we obtained

$$d_t\Phi=\frac{1}{2}(\boldsymbol{V}_{\rm a}^2-\boldsymbol{V}_{\rm e}^2)-H-\phi\,. \tag{21}$$

The right–hand side of Eq. (21) is called the kinetic potential. For barotropic atmosphere, it reduces to the kinetic potential defined in Eq. (5) of ER49. Because the quantities α and β are time invariant, by using Eq. (6) and performing similar steps as those in Eq. (12), we obtain

$$\begin{aligned}&d_t\left[\left(\frac{\boldsymbol{\xi}+2\boldsymbol{\Omega}-\nabla\eta\times\nabla S}{\rho}\right)\cdot\alpha\nabla\beta\right]\\&=\alpha\left(\frac{\boldsymbol{\xi}+2\boldsymbol{\Omega}-\nabla\eta\times\nabla S}{\rho}\right)\nabla d_t\beta=0\,.\end{aligned} \tag{22}$$

This leads to a new invariant, which is given by

$$\mathrm{GER}=\boldsymbol{\xi}_{\rm g}\cdot\boldsymbol{V}_{\rm g}/\rho\,, \tag{23}$$

and

$$d_t\mathrm{GER}/dt=0\,, \tag{24}$$

where $\boldsymbol{\xi}_{\rm g}=\boldsymbol{\xi}_{\rm a}-\nabla\eta\times\nabla S$ is the generalized vorticity as defined in Eq. (6), and $\boldsymbol{V}_{\rm g}=\alpha\nabla\beta=\boldsymbol{V}_{\rm a}-\nabla\Phi-\eta\nabla S$ [see Eq. (18)] is the generalized velocity.

Note that $\boldsymbol{\xi}_{\rm g}$ is the generalized vorticity, while $\boldsymbol{V}_{\rm g}$ is the generalized velocity. Hence, $\boldsymbol{\xi}_{\rm g}\cdot\boldsymbol{V}_{\rm g}$ has the same physical dimension as the helicity $\boldsymbol{\xi}\cdot\boldsymbol{V}$, which may be called the "generalized helicity". The similar definition can be found in Gaffet (1985) and Mobbs (1981) in both the barotropic and baroclinic atmosphere. However, Gaffet (1985) obtained the helicity under the restriction of zero potential vorticity flow, and Mobbs (1981) gave the generalized helicity on the basis of the Weber's transformation and did not include the gradient of Lagrangian–time integrated kinetic potential.

The new invariant

$$\mathrm{GER}=\frac{\boldsymbol{\xi}+2\boldsymbol{\Omega}-\nabla\eta\times\nabla s}{\rho}\cdot(\boldsymbol{V}_{\rm a}-\nabla\Phi-\eta\nabla s)\,, \tag{25}$$

in non–rotating frame becomes

$$\mathrm{GER}'=\frac{\boldsymbol{\xi}-\nabla\eta\times\nabla s}{\rho}\cdot(\boldsymbol{V}-\nabla\Phi'-\eta\nabla s)\,, \tag{26}$$

where

$$d_t\Phi'=\frac{\boldsymbol{V}^2}{2}-\phi-H\,.$$

GER' is the same as the baroclinic Ertel–Rossby invariant obtained in ER03, which is deduced by the Weber transformation. The relationship between the new GER invariant and the traditional potential vorticity and the generalized vorticity were not given in this study.

From Eq. (25), we have

$$\begin{aligned}d_t(\nabla\Phi\cdot\boldsymbol{\xi}_{\rm g})&=\boldsymbol{\xi}_{\rm g}\cdot\nabla d_t\Phi\\&=\left[\frac{1}{2}\nabla(\boldsymbol{V}_{\rm a}^2-\boldsymbol{V}_{\rm e}^2)-\nabla\phi-\nabla H\right]\cdot\boldsymbol{\xi}_{\rm g}\\&=\frac{1}{2}\nabla(\boldsymbol{V}_{\rm a}^2-\boldsymbol{V}_{\rm e}^2)\cdot\boldsymbol{\xi}_{\rm g}-\\&\quad(\nabla\phi+1/\rho\nabla p)\cdot\boldsymbol{\xi}_{\rm g}-T\nabla s\cdot\boldsymbol{\xi}_{\rm g}\,.\end{aligned} \tag{27}$$

Combining Eqs. (27) and (25) obtains

$$\begin{aligned}d_t\mathrm{GER}&=d_t\left(\boldsymbol{V}_{\rm a}\cdot\boldsymbol{\xi}_{\rm g}\right)-\frac{1}{2}\nabla(\boldsymbol{V}_{\rm a}^2-\boldsymbol{V}_{\rm e}^2)\cdot\\&\quad\boldsymbol{\xi}_{\rm g}+(\nabla\phi+\frac{1}{\rho}\nabla p)\cdot\boldsymbol{\xi}_{\rm g}\\&=d_t\{\xi_{\rm g}\cdot(\boldsymbol{V}_{\rm a}-\nabla W)\}\\&=0,\end{aligned} \tag{28}$$

where

$$\frac{dW}{dt}=\frac{1}{2}(\boldsymbol{V}_a^2-\boldsymbol{V}_e^2)-\phi-\int\frac{dp}{\rho}\,. \tag{29}$$

It leads to another form of the GER as

$$\mathrm{GER}'=\frac{\xi+2\boldsymbol{\Omega}-\nabla\eta\times\nabla s}{\rho}\cdot(\boldsymbol{V}_{\rm a}-\nabla W)\,. \tag{30}$$

For barotropic atmosphere, Eq. (30) reduces to

$$\mathrm{GER}'=\frac{\boldsymbol{\xi}_{\rm a}}{\rho}\cdot(\boldsymbol{V}_{\rm a}-\nabla W)\,, \tag{31}$$

and this gives the traditional Ertel–Rossby invariant.

4. Concluding remarks

In this study, the generalized Ertel–Rossby (GER) invariant, was introduced physically and derived mathematically on the basis of the generalized vorticity equation and the generalized velocity equation. The generalized vorticity equation provided a method to find new invariants, and the velocity equation which was obtained by Clebsch transformation was used to give the tracer function. The new invariant GER was expressed by the dot product of the generalized vorticity and the generalized velocity. The generalized vorticity was the absolute vorticity minus the cross–product of the gradient of Lagrangian–time integrated temperature and the gradient of entropy. The generalized velocity was the absolute velocity minus the sum of the gradient of Lagrangian–time integrated kinetic potential and the Lagrangian–time integrated temperature multiplied by the gradient of entropy. Because GER contains much different dynamic information from the traditional potential vorticity, it may provide another useful tool to study the atmospheric dynamic processes for phenomena ranging from large scales to small scales.

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